

MAXIMAL SUBALGEBRAS OF MATRIX LIE SUPERALGEBRAS

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ABSTRACT. Dynkin's classification of maximal subalgebras of simple finite dimensional complex Lie algebras is generalized to matrix Lie superalgebras, i.e., the Lie subsuperalgebras of $\mathfrak{gl}(p|q)$.

INTRODUCTION

0.1. Dynkin's result. In 1951 Dynkin published two remarkable papers [D1], [D2], somewhat interlaced in their theme: classification of semisimple subalgebras of simple (finite dimensional) Lie algebras and classification of maximal subalgebras of simple (finite dimensional) Lie algebras. These classifications are of interest *per se*; they also proved to be useful in the studies of integrable systems and in representation theory.

According to the general ideology of Leites' *Seminar on supermanifolds* we have to generalize Dynkin's results to "classical" Lie superalgebras, i.e., to Lie algebras of polynomial growth "close" in a sense to simple ones. This problem was raised by A. L. Onishchik and D. Leites. In this paper (partly preprinted long ago [Sh1]) I try to give the answer in a form similar to that of Dynkin's result.

Let me first remind Dynkin's result. Let \mathfrak{g} be a simple matrix Lie algebra over \mathbb{C} , i.e., $\mathfrak{sl}(n)$, $\mathfrak{o}(n)$ or $\mathfrak{sp}(2n)$. In what follows a "matrix" Lie (super)algebra means that the elements of the Lie (super)algebra in question is realized by matrices. Let V be the space of the standard (identity) representation id of \mathfrak{g} in the space of column-vectors. Let $\mathfrak{h} \subset \mathfrak{g}$ be a maximal subalgebra. Then only the following 3 cases might occur:

- 1) the representation of \mathfrak{h} in V is reducible (we say that \mathfrak{h} is reducible);
- 2) \mathfrak{h} is simple and irreducible (i.e., \mathfrak{h} irreducibly acts on V);
- 3) \mathfrak{h} is irreducible but not simple.

The answers in these cases are:

1) \mathfrak{h} can be described as the collection of all operators from \mathfrak{g} that preserve a subspace $W \subset V$. Here W can be arbitrary for $\mathfrak{g} = \mathfrak{sl}$ case while for $\mathfrak{g} = \mathfrak{o}(n)$ and $\mathfrak{sp}(2n)$ the bilinear form B on V preserved by \mathfrak{g} must be either nondegenerate or identically zero on W .

2) For practically every irreducible representation of a simple \mathfrak{h} the image is a maximal subalgebra in a simple matrix algebra and Dynkin listed the exceptions.

3) Here is the complete list of nonsimple *maximal irreducible subalgebras* (abbreviation for *maximal subalgebras which are irreducible*):

$$\begin{array}{llllll}
 \mathfrak{sl}(V_1) \oplus \mathfrak{sl}(V_2) & \text{in} & \mathfrak{sl}(V_1 \otimes V_2) & \text{if} & \dim V_2 \geq \dim V_1 \geq 2 & \\
 \mathfrak{sp}(V_1) \oplus \mathfrak{o}(V_2) & \text{in} & \mathfrak{sp}(V_1 \otimes V_2) & \text{if} & \dim V_1 \geq 2, \dim V_2 \geq 3 & \\
 & & & & \text{and } \dim V_2 \neq 4 \text{ or} & \\
 & & & & \text{if } \dim V_1 = 2 \text{ and } \dim V_2 = 4; & \\
 \mathfrak{sp}(V_1) \oplus \mathfrak{sp}(V_2) & & & \text{if} & \dim V_2 \geq \dim V_1 > 2 & (*) \\
 \text{or} & & & & & \\
 & \text{in} & \mathfrak{o}(V_1 \otimes V_2) & & & \\
 \mathfrak{o}(V_1) \oplus \mathfrak{o}(V_2) & & & \text{if} & \dim V_2 \geq \dim V_1 \geq 3 & \\
 & & & & \text{and } \dim V_1, \dim V_2 \neq 4. &
 \end{array}$$

Difficulties we encounter in superization of Dynkin's results 1)–3) range widely:

Superization of 1) to Lie superalgebras with Cartan matrix seem to be straightforward. Observe, however, that among classical Lie superalgebras, as well as among infinite dimensional Lie algebras, we naturally encounter algebras which can not be described with the help of Cartan matrix. Such algebras possess, nevertheless, a system of roots that can be split into the disjoint union of positive and negative ones and the

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notion of a simple root is well-defined. For a description of systems of simple roots see [PS], [DP]. The corresponding generators are referred to as *Chevalley generators*. In the absence of Cartan matrix generators do not quite correspond to the simple-root vectors, see [LSH1]; they are called *generalized Chevalley generators*. Clearly, *parabolic* subalgebras — distinguished by just one missing generalized Chevalley generator — are maximal. Contrary to the naive expectations they do not exhaust all maximal reducible subalgebras even in matrix Lie superalgebras considered in [ZZO], where a minor mistake of [D2] is corrected in passing. We will consider this case elsewhere [LSH2].

Superization of 2) was impossible until recently. For a partial result see [J1]. Now that Penkov and Serganova [PS] derived the character formula for any irreducible finite dimensional representation of a finite dimensional Lie superalgebra we can consider this problem and intend to do it elsewhere.

Superization of 3) is what is done in this paper: the description the irreducible nonsimple maximal subsuperalgebras of matrix Lie superalgebras either simple or “classical” — certain algebras closely related to simple ones.

Regrettably, a precise description of the maximal Lie superalgebras of type 3) is more involved than (already not very concise) Dynkin’s description (*) above. Therefore, in order to grasp the formulation, we split it into several statements (Theorems 0.2.1–0.2.6). Still, a sufficiently transparent and comparable with (*) Main Theorem (below) singles out a small amount of contenders for the role of maximal subalgebras; these contenders almost always are, indeed, maximal.

To make the paper readable we recall the list of classical matrix Lie superalgebras and the general background on superspaces (see Appendix 2). Assuming the reader is familiar with these notions, let us formulate the result explicitly.

0.2. Our result. Throughout the paper the ground field is \mathbb{C} . In order to formulate our result, we need several notations listed under bullets.

- Given two representations of Lie superalgebras $\mathfrak{g}_i \rightarrow \mathfrak{gl}(V_i)$ we get a representation of the Lie superalgebra $\mathfrak{g}_1 \oplus \mathfrak{g}_2 \rightarrow \mathfrak{gl}(V_1 \otimes V_2)$ in the superspace $V = V_1 \otimes V_2$, called the *tensor product* of the given representations, setting

$$g_1 + g_2 \mapsto g_1 \otimes 1 + 1 \otimes g_2 \quad \text{for } g_1 \in \mathfrak{g}_1, g_2 \in \mathfrak{g}_2. \quad (0.2)$$

If both \mathfrak{g}_1 and \mathfrak{g}_2 contain the identity operators, the representation (0.2) has a 1-dimensional kernel. By $\mathfrak{g}_1 \odot \mathfrak{g}_2$ we will mean the image of the direct sum under the representation (0.2) if not both the \mathfrak{g}_i are of type \mathfrak{q} . It is convenient to retain the notation $\mathfrak{g}_1 \odot \mathfrak{g}_2$ even in the absence of the kernel, i.e., when $\mathfrak{g}_1 \odot \mathfrak{g}_2 = \mathfrak{g}_1 \oplus \mathfrak{g}_2$.

- If both the \mathfrak{g}_i are of type \mathfrak{q} the construction of $\mathfrak{g}_1 \odot \mathfrak{g}_2$ is different. Namely, let V_0 be a $(1, 1)$ -dimensional superspace and $\mathcal{A} \subset \text{Mat}(1|1) = \text{Mat}(V_0)$ the associative superalgebra of matrices of the form $\begin{pmatrix} a & b \\ b & a \end{pmatrix}$, where $a, b \in \mathbb{C}$. Then the *associative* superalgebra $C(\mathcal{A})$ — the centralizer of \mathcal{A} — consists of matrices of the form $\begin{pmatrix} c & d \\ -d & c \end{pmatrix}$, where $c, d \in \mathbb{C}$. Clearly, $\mathcal{A} \cong C(\mathcal{A})$.

Observe that the Lie superalgebra $\mathfrak{q}(n)$ can be represented as the tensor product $\mathfrak{gl}(n|0) \otimes \mathcal{A}$ of the Lie algebra $\mathfrak{gl}(n|0)$ by the associative superalgebra \mathcal{A} . Then the space W of the standard representation of $\mathfrak{q}(n)$ can be represented as $W = W_0 \otimes \mathcal{A}$ in a natural way:

$$(g \otimes a_1)(w \otimes a_2) = g \cdot w \otimes a_1 a_2 \quad \text{for any } g \in \mathfrak{gl}(n|0), w \in W_0, a_1, a_2 \in \mathcal{A}.$$

Let now a superspace V be represented as $V = V_1 \otimes \mathcal{A} \otimes V_2$, where $\dim V_1 = (n_1, 0)$, $\dim V_2 = (n_2, 0)$. Then a natural action ρ of $\mathfrak{q}(n_1) \oplus \mathfrak{q}(n_2)$ is determined in V : indeed, $\mathfrak{q}(n_1)$ acts in $V_1 \otimes \mathcal{A}$, and $\mathfrak{q}(n_2)$ acts in $\mathcal{A} \otimes V_2 \cong C(\mathcal{A}) \otimes V_2$.

In terms of matrices this action takes the form:

$$\begin{pmatrix} A & B \\ B & A \end{pmatrix} \in \mathfrak{q}(n_1) \mapsto \begin{pmatrix} A \otimes 1 & B \otimes 1 \\ B \otimes 1 & A \otimes 1 \end{pmatrix} \in \mathfrak{gl}(V_1 \otimes \mathcal{A});$$

$$\begin{pmatrix} C & D \\ D & C \end{pmatrix} \in \mathfrak{q}(n_2) \mapsto \begin{pmatrix} 1 \otimes C & i \otimes D \\ -i \otimes D & 1 \otimes C \end{pmatrix} \in \mathfrak{gl}(\mathcal{A} \otimes V_2).$$

The representation ρ has a one-dimensional kernel. Denote the quotient of $\mathfrak{q}(n_1) \oplus \mathfrak{q}(n_2)$ modulo this kernel by $\mathfrak{q}(n_1) \odot \mathfrak{q}(n_2)$.

- We often use a general notation $\mathbf{aut}(B)$ for the Lie superalgebra that preserves the bilinear form B . The Lie superalgebra $\mathbf{aut}(B)$ turns into either \mathbf{osp} or \mathbf{pe} depending on the parity of B .

- Denote by $\mathbf{hei}(0|2n)$ the Heisenberg Lie superalgebra with $2n$ odd generators of creation and annihilation, i.e., the Lie superalgebra with odd generators $\xi_1, \dots, \xi_n; \eta_1, \dots, \eta_n$ and an even generator z satisfying the

relations

$$[\xi_i, \eta_j] = \delta_{i,j} \cdot z, \quad [\xi_i, \xi_j] = [\eta_i, \eta_j] = [z, \mathfrak{hei}(0|2n)] = 0.$$

The only irreducible finite dimensional representation of $\mathfrak{hei}(0|2n)$ that sends z into the identity operator is realized on the superspace $\Lambda(n) = \Lambda(\xi)$ by the formulas $\xi_i \mapsto \xi_i$, $\eta_i \mapsto \partial_i = \frac{\partial}{\partial \xi_i}$.

The normalizer of $\mathfrak{hei}(0|2n)$ in $\mathfrak{gl}(\Lambda(n))$ is $\mathfrak{g} = \mathfrak{hei}(0|2n) \oplus \mathfrak{o}(2n)$; we can realize \mathfrak{g} in the space of the spinor representation of $\mathfrak{o}(2n)$ by the following differential operators: $\mathfrak{g} = \text{Span}(1; \xi, \partial; \xi\xi, \xi\partial, \partial\partial)$.

• Define the representation $T^{1/2}$ of $\mathfrak{vect}(0|n)$ in the superspace $\Lambda(\xi)\sqrt{\text{vol}(\xi)}$ of halfdensities by the formula $T^{1/2}(D) = D + \frac{1}{2}\text{div}D$ and define the form ω on $\Lambda(\xi)\sqrt{\text{vol}}$ by the formula $\omega(f\sqrt{\text{vol}}, g\sqrt{\text{vol}}) = \int fg \cdot \text{vol}(\xi)$.

Main Theorem . 1° Let \mathfrak{g} be an irreducible matrix Lie superalgebra which is neither almost simple nor a central extension of an almost simple Lie superalgebra.

Then \mathfrak{g} is contained in one of the following four major types of Lie superalgebras and only in them:

- 1) $\mathfrak{gl}(V_1) \odot \mathfrak{gl}(V_2)$;
- 2) $\mathfrak{q}(V_1) \odot \mathfrak{q}(V_2)$;
- 3) $\mathfrak{gl}(V) \otimes \Lambda(n) \oplus \mathfrak{vect}(0|n)$;
- 4) $\mathfrak{hei}(0|2n) \oplus \mathfrak{o}(2n)$.

2° Let in addition to conditions of 1° , \mathfrak{g} be a subalgebra of $\mathfrak{q}(V) = C(J)$. Then \mathfrak{g} is contained in one of the following Lie superalgebras (numbered as in 1°) and only in them:

- 1q) $\mathfrak{q}(V_1) \odot \mathfrak{gl}(V_2)$ and $J = J_1 \otimes 1$, where $\mathfrak{q}(V_1) = C(J_1)$;
- 3q) $\mathfrak{q}(V_0) \otimes \Lambda(n) \oplus \mathfrak{vect}(0|n)$ and $J = J_0 \otimes 1$, where $\mathfrak{q}(V_0) = C(J_0)$;

3° Let, in addition to conditions of 1° , \mathfrak{g} preserve a nondegenerate homogeneous form ω , either symmetric or skew-symmetric. Then \mathfrak{g} is contained in one of the following Lie superalgebras (numbered as in 1°) and only in them:

- 1 ω) $\mathfrak{aut}(\omega_1) \odot \mathfrak{aut}(\omega_2)$ and $\omega = \omega_1 \otimes \omega_2$;
- 3 ω) $T^{1/2}(\mathfrak{vect}(0|n))$, where ω is the form on half-densities.

The union of the following statements 0.2.1–0.2.6 constitute, more or less, a statement converse to Main Theorem. The basic types of examples considered in these theorems appear in §1 labelled 1.1 – 1.4.

0.2.1. Theorem . Let $\dim V_i = (n_i, m_i)$ and $(n_i, m_i) \neq (1, 1), (1, 0), (0, 1)$. Then

- 1) $\mathfrak{gl}(V_1) \odot \mathfrak{gl}(V_2)$ is maximal irreducible in $\mathfrak{gl}(V)$ if either $n_1 \neq m_1$ or $n_2 \neq m_2$;
- 2) the superalgebras $\mathfrak{sl}(V_1) \oplus \mathfrak{sl}(V_2)$ if either $n_1 \neq m_1$ or $n_2 \neq m_2$ and $\mathfrak{gl}(V_1) \odot \mathfrak{gl}(V_2)$ if $n_1 = m_1$ and $n_2 = m_2$ are maximal irreducible in $\mathfrak{sl}(V)$.

0.2.2. Let $\dim V_1 = (m, m)$; let V_2 be of any (finite) superdimension. For odd operator $J_1 \in \text{End}(V_1)$ such that $J_1^2 = -1_{V_1}$, we clearly see that $J = J_1 \otimes 1$ satisfies $J^2 = -1_V$. Set $\mathfrak{q}(V_1) = C(J_1)$ and $\mathfrak{q}(V) = C(J)$.

Theorem . The Lie superalgebra $\mathfrak{q}(V_1) \odot \mathfrak{gl}(V_2)$ is maximal irreducible in $\mathfrak{q}(V)$.

0.2.3. Let $\mathfrak{g}_1 = \mathfrak{q}(V_1) = \mathfrak{q}(n_1) = C(J_1)$ and $\mathfrak{g}_2 = \mathfrak{q}(V_2) = \mathfrak{q}(n_2) = C(J_2)$. The representation ρ of $\mathfrak{g}_1 \oplus \mathfrak{g}_2$ in $V = V_1 \otimes V_2$ has, clearly, the 2|2-dimensional Lie superalgebra of intertwining operators spanned by

$$\text{id} = 1_V, \quad J_1 \otimes 1, \quad 1 \otimes J_2, \quad J_1 \otimes J_2.$$

Set $J = J_1 \otimes J_2$. Clearly, $J^2 = -1_V$; the restrictions $\rho_{\pm i}$ of ρ to the eigenspaces $V_{\pm i}$ of J are irreducible and the change of parity functor Π permutes V_i with V_{-i} .

Theorem . The images $\rho_{\pm i}(\mathfrak{q}(V_1) \oplus \mathfrak{q}(V_2))$ are isomorphic to $\mathfrak{q}(V_1) \odot \mathfrak{q}(V_2)$. They are maximal irreducible in $\mathfrak{sl}(V_{\pm i})$.

0.2.4. Let us consider Lie superalgebras $\mathfrak{g}_i = \mathfrak{aut}(\omega_i)$ each preserving a bilinear form ω_i in the superspace V_i ; let B_i be the matrices of these forms, $i = 1, 2$. Clearly, in $V = V_1 \otimes V_2$ the form $\omega = \omega_1 \otimes \omega_2$ of parity $p(\omega_1) + p(\omega_2)$ arises; if the ω_i are nondegenerate, then so is ω . Let $\mathfrak{g} = \mathfrak{aut}(\omega)$. Formula (0.2) defines a representation of $\mathfrak{g}_1 \oplus \mathfrak{g}_2$ in \mathfrak{g} .

Theorem . The Lie superalgebra $\mathfrak{g}_1 \oplus \mathfrak{g}_2$ is maximal irreducible in \mathfrak{g} if $p(\omega) = \bar{0}$ or if $p(\omega_1) = \bar{0}$, $p(\omega_2) = \bar{1}$ and $\dim V_1 \neq (n, n)$.

If $p(\omega_1) = \bar{0}$, $p(\omega_2) = \bar{1}$ and $\dim V_1 = (n, n)$, then $\mathfrak{g}_1 \oplus \mathfrak{g}_2$ is maximal irreducible in $\mathfrak{g} \cap \mathfrak{sl}(V)$.

More exactly, the following subalgebras $\mathfrak{g}_1(V_1) \oplus \mathfrak{g}_2(V_2)$ are maximal in $\mathfrak{g}(V_1 \otimes V_2)$:

\mathfrak{g}_1	\mathfrak{g}_2	\mathfrak{g}
$\mathfrak{osp}(n_1 2m_1)$	$\mathfrak{osp}(n_2 2m_2)$	$\mathfrak{osp}(n_1n_2 + 4m_1m_2 2n_1m_2 + 2n_2m_1)$
$\mathfrak{o}(n)$	$\mathfrak{osp}(n_2 2m_2)$	$\mathfrak{osp}(nn_2 2nm_2), n \neq 2, 4$
$\mathfrak{sp}(2n)$	$\mathfrak{osp}(n_2 2m_2)$	$\mathfrak{osp}(2mn_2 4nm_2)$
$\mathfrak{pe}(n_1)$	$\mathfrak{pe}(n_2)$	$\mathfrak{osp}(2n_1n_2 2n_1n_2), n_1, n_2 > 2$
$\mathfrak{osp}(n_1 2m_1)$	$\mathfrak{pe}(n_2)$	$\mathfrak{pe}(n_1n_2 + 2m_1n_2)$ if $n_1 \neq 2m_1$ $\mathfrak{spe}(n_1n_2 + 2m_1n_2)$ if $n_1 = 2m_1$
$\mathfrak{o}(n)$	$\mathfrak{pe}(m)$	$\mathfrak{pe}(nm)$
$\mathfrak{sp}(2n)$	$\mathfrak{pe}(m)$	$\mathfrak{pe}(2nm)$

0.2.5. The maximal subalgebras considered in sec. 0.2.1–0.2.4 are similar to those considered by Dynkin. There are, however, maximal subalgebras of matrix superalgebras of totally different nature.

Let V_1 be a linear superspace of dimension (r, s) ; let $\Lambda(n)$ the Grassmann superalgebra with n odd generators ξ_1, \dots, ξ_n and $\mathfrak{vect}(0|n) = \mathfrak{der}\Lambda(n)$ the Lie superalgebra of vector fields on the $(0, n)$ -dimensional supermanifold.

Let $\mathfrak{g} = \mathfrak{gl}(V_1) \otimes \Lambda(n) \oplus \mathfrak{vect}(0|n)$ be the semidirect product with the natural action of $\mathfrak{vect}(0|n)$ on the ideal $\mathfrak{gl}(V_1) \otimes \Lambda(n)$. The Lie superalgebra \mathfrak{g} has a natural faithful representation ρ in the space $V = V_1 \otimes \Lambda(n)$ defined by the formulas

$$\begin{aligned}\rho(X \otimes \varphi)(v \otimes \psi) &= (-1)^{p(\varphi)p(\psi)} Xv \otimes \varphi\psi; \\ \rho(D)(v \otimes \psi) &= -(-1)^{p(D)p(v)} v \otimes D\psi;\end{aligned}$$

for any $X \in \mathfrak{gl}(V_1)$, $\varphi, \psi \in \Lambda(n)$, $v \in V_1$, $D \in \mathfrak{vect}(0|n)$.

Let us identify the elements from \mathfrak{g} with their images under ρ , so we consider \mathfrak{g} embedded into $\mathfrak{gl}(V)$.

Theorem . 1) *The Lie superalgebra $\mathfrak{gl}(V_1) \otimes \Lambda(n) \oplus \mathfrak{vect}(0|n)$ is maximal irreducible in $\mathfrak{sl}(V_1 \otimes \Lambda(n))$ unless a) $\dim V_1 = (1, 1)$ or b) $n = 1$ and $\dim V_1 = (1, 0)$ or $(0, 1)$ or (r, s) for $r \neq s$.*

2) *If $\dim V_1 = (1, 1)$, then $\mathfrak{gl}(1|1) \cong \Lambda(1) \oplus \mathfrak{vect}(0|1)$, so $\mathfrak{gl}(V_1) \otimes \Lambda(n) \oplus \mathfrak{vect}(0|n) \subset \Lambda(n+1) \oplus \mathfrak{vect}(0|n+1)$ and it is the latter superalgebra which is maximal irreducible in $\mathfrak{sl}(V)$.*

3) *If $n = 1$ and $\dim V_1 = (r, s)$ for $r > s > 0$, then \mathfrak{g} is maximal irreducible in $\mathfrak{gl}(V)$.*

0.2.5'. Let $\mathfrak{g}_1 = \mathfrak{q}(n_1) = \mathfrak{q}(V_1) = C(J_1)$; let $V \cong V_1 \otimes \Lambda(n)$ and $J = J_1 \otimes \text{id}_{\Lambda(n)}$. Clearly, $J^2 = -\text{id}_V$. It is also clear that

$$\mathfrak{g}_1 \otimes \Lambda(n) \oplus \mathfrak{vect}(0|n) \subset \mathfrak{q}(V) = C(J).$$

Theorem . *The Lie superalgebra $\mathfrak{q}(n_1) \otimes \Lambda(n) \oplus \mathfrak{vect}(0|n)$ is maximal irreducible in $\mathfrak{q}(V_1 \otimes \Lambda(n))$.*

0.2.6. Theorem . 1) *For $n \geq 2$ and $n \neq 3$ the algebra $\mathfrak{g} = \mathfrak{hei}(0|2n) \oplus \mathfrak{o}(2n)$ is maximal irreducible in $\mathfrak{sl}(\Lambda(n))$.*

2) *For $n = 3$ the algebra \mathfrak{g} is contained in the nontrivial central extension \mathfrak{as} of $\mathfrak{spe}(4)$; the Lie superalgebra \mathfrak{as} is maximal irreducible in $\mathfrak{sl}(\Lambda(3))$.*

0.3. Realted results.

0.3.1. Comparison with the case of prime characteristic. Our Theorems 0.2.5 and 0.2.6 are very similar to the result in prime characteristic announced by A. I. Kostrikin's student O. K. Ten.

Recall that a subalgebra of a (finite dimensional) Lie algebra is called *regular* if it is invariant with respect to a maximal torus.

Theorem . ([T]) *Let \mathbb{K} be an algebraically closed field of characteristic $p > 3$.*

1) *Any non-semisimple maximal subalgebra of $\mathfrak{sl}(n)$ for $n \not\equiv 0 \pmod{p}$, $\mathfrak{o}(n)$ or $\mathfrak{sp}(2n)$ is regular.*

2a) *Let $\dim V = np^m$, $(n, p) = 1$, $n > 1$. If \mathfrak{m} is an irreducible maximal subalgebra in $\mathfrak{sl}(V)$ such that $\mathfrak{pm} = \mathfrak{m}/\langle 1 \rangle$ is not semisimple, then $V = U \otimes O_m$, where $O_m = \mathbb{K}[x_1, \dots, x_m]/(x_1^p, \dots, x_m^p)$ and $\mathfrak{m} = \mathfrak{gl}(U) \otimes O_m \oplus \mathfrak{vect}_m$.*

2b) *If $n = 1$, then in addition to the above examples 2a) the algebra $\mathfrak{m} = \mathfrak{sp}_{2m} \oplus \mathfrak{hei}(2m|0)$ is also maximal in $\mathfrak{sl}(V)$.*

3) *Any maximal subalgebra in \mathfrak{g}_2 is regular except \mathfrak{vect}_1 for $p = 7$ and $\mathfrak{sl}(2)$ for $p > 7$.*

0.3.2. The maximal (almost) simple subalgebras of simple finite dimensional Lie superalgebras.

It seems that it is only possible to describe such subalgebras when they are simple and, moreover, are isomorphic to $\mathfrak{osp}(1|2n)$, whose finite dimensional representations are completely reducible. Perhaps, it is possible to extend the result to $\mathfrak{osp}(2|2n)$ and $\mathfrak{sl}(1|n)$, whose finite dimensional representations are “tame” in a sense (see [L4]) and, therefore, describable.

- Embeddings of $\mathfrak{osp}(1|2)$ are described by Serganova in [LSS]. (Partly, her results were independently rediscovered by J. Van der Jeugt [J2], where *regular* Lie subsuperalgebras of finite dimensional simple Lie superalgebras with Cartan matrix are described, cf. [LSH2].)

- A subalgebra \mathfrak{h} of the Lie superalgebra \mathfrak{g} is called *Volichenko algebra* if it is not a Lie subsuperalgebra. A list of simple finite dimensional Volichenko subalgebras in simple Lie superalgebras is obtained under a technical condition by Serganova [S]. (For motivations and infinite dimensional case see [LS] and [KL].)

0.3.3. Nonsimple maximal subalgebras. Maximal *solvable* Lie superalgebras of $\mathfrak{gl}(m|n)$ and $\mathfrak{sl}(m|n)$ are classified in [Sh2]. A strange series of subalgebras was discovered.

0.3.4. Maximal subalgebras of nonmatrix superalgebras. In the case when the ambient possesses a Cartan matrix this is done in [J2] with a gap filled in by Serganova (published in [Sh3]). For the classification of graded maximal subalgebras of vectorial Lie superalgebras see [Sh1] and [Sh3].

§1. IRREDUCIBLE NON-SIMPLE MATRIX MAXIMAL SUBALGEBRAS OF $\mathfrak{gl}(m|n)$ AND $\mathfrak{sl}(m|n)$

The 4 basic types of examples of maximal subalgebras from Main Theorem are considered in 1.1 – 1.4.

1.1. Theorem . Let $\dim V_i = (n_i, m_i)$ and $(n_i, m_i) \neq (1, 1), (1, 0), (0, 1)$. Let $V = V_1 \otimes V_2$. Then

1) $\mathfrak{gl}(V_1) \odot \mathfrak{gl}(V_2)$ is a maximal subalgebra in $\mathfrak{gl}(V)$ if $n_1 \neq m_1$ or $n_2 \neq m_2$; it is maximal in $\mathfrak{sl}(V)$ otherwise.

2) The following subalgebras are maximal in $\mathfrak{sl}(V)$: a) $\mathfrak{sl}(V_1) \oplus \mathfrak{sl}(V_2)$ if $n_1 \neq m_1$ or $n_2 \neq m_2$; b) $\mathfrak{gl}(V_1) \odot \mathfrak{gl}(V_2)$ if $n_1 = m_1$ and $n_2 = m_2$.

Proof. 1) The space of all operators in an arbitrary superspace V may be endowed with two structures: that of the Lie superalgebra $\mathfrak{gl}(V)$ and of the associative superalgebra $\text{Mat}(V)$. Observe that $\text{Mat}(V_1) \otimes \text{Mat}(V_2) \cong \text{Mat}(V_1 \otimes V_2)$. Therefore, any element $g \in \mathfrak{gl}(V_1 \otimes V_2)$ can be represented in the form $g = \sum A_i \otimes B_i$, where $A_i \in \mathfrak{gl}(V_1)$, $B_i \in \mathfrak{gl}(V_2)$. The bracket in $\mathfrak{gl}(V_1 \otimes V_2)$ is defined via

$$[A \otimes B, C \otimes D] = (-1)^{p(B)p(C)}[A, C] \otimes BD + (-1)^{p(A \otimes B)p(C)}CA \otimes [B, D]. \quad (1.1)$$

Now let

$$\mathfrak{g} = \mathfrak{gl}(V_1) \odot \mathfrak{gl}(V_2) = \{A \otimes 1 + 1 \otimes B\}, \quad \text{where } A \in \mathfrak{gl}(V_1), B \in \mathfrak{gl}(V_2),$$

and let \mathfrak{f} be an intermediate subalgebra, i.e.,

$$\mathfrak{g} \subset \mathfrak{f} \subset \mathfrak{gl}(V) \quad (1.2)$$

Let $h \in \mathfrak{f}$ be an element which is not contained in \mathfrak{g} . Then $h = \sum A_i \otimes B_i$, where $A_i \in \mathfrak{gl}(V_1)$, $B_i \in \mathfrak{gl}(V_2)$. Thanks to (1.2) \mathfrak{f} contains all elements of the form $[g, h]$, $g \in \mathfrak{g}$. Therefore, due to (1.1), we have

$$[A \otimes 1, h] = \sum [A, A_i] \otimes B_i, \quad [1 \otimes B, h] = \sum (-1)^{p(B)p(A_i)} A_i \otimes [B, B_i].$$

Since A and B are arbitrary operators of $\mathfrak{gl}(V_1)$ and $\mathfrak{gl}(V_2)$ respectively, \mathfrak{f} contains h and all the elements of the form $h_C = \sum C_i \otimes B_i$ and $h_D = \sum A_i \otimes D_i$, where C_i (resp. D_i) though related perhaps, run over an ideal $I_i \subset \mathfrak{gl}(V_1)$ (resp. $J_i \subset \mathfrak{gl}(V_2)$) for each fixed i .

Now, observe that,

1) $\mathfrak{gl}(W)$ contains only two ideals, namely, $I_0 = \mathbb{C} \cdot 1$ and $I_1 = \mathfrak{sl}(W)$ for any W of dimension not equal to $(1, 1)$, $(1, 0)$, $(0, 1)$ and

2) \mathfrak{f} contains the subalgebras $\mathfrak{gl}(V_1) \otimes 1$ and $1 \otimes \mathfrak{gl}(V_2)$.

This implies that \mathfrak{f} contains the set $\mathfrak{sl}(V_1) \otimes \mathfrak{sl}(V_2)$. Since the set $\mathfrak{sl}(W)$ is not an associative subalgebra of $\text{Mat}(W)$, then (1.1) implies that

$$\mathfrak{f} \supset \mathfrak{sl}(V_1) \otimes \text{Mat}(V_2) \oplus \text{Mat}(V_1) \otimes \mathfrak{sl}(V_2) = \mathfrak{sl}(V_1 \otimes V_2).$$

If, in addition, $n_1 \neq m_1$, then $\text{str}(1 \otimes B) = (n_1 - m_1)\text{str}B \neq 0$, if $\text{str}B \neq 0$, i.e., \mathfrak{f} coincides with the whole $\mathfrak{gl}(V)$.

If $n_1 = m_1$ and $n_2 = m_2$, then $\mathfrak{gl}(V_1) \odot \mathfrak{gl}(V_2)$ is contained in $\mathfrak{sl}(V)$ and by the above arguments is a maximal Lie subsuperalgebra of $\mathfrak{sl}(V)$.

To conclude the proof of heading 2a), it only suffices to notice that $1 \otimes B \in \mathfrak{sl}(V)$ if and only if $B \in \mathfrak{sl}(V_2)$ for $n_1 \neq m_1$ and $B \in \mathfrak{gl}(V_2)$ for $n_1 = m_1$. \square

1.2. Let W_1 and W_2 be standard representations of $\mathfrak{g}_1 = \mathfrak{q}(n) = C(J_1)$ and $\mathfrak{g}_2 = \mathfrak{q}(m) = C(J_2)$, respectively. The representation ρ of $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ in the tensor product $V = V_1 \otimes V_2$ has the $(2, 2)$ -dimensional algebra of intertwining operator $\langle 1 \otimes 1, J_1 \otimes 1, 1 \otimes J_2, J_1 \otimes J_2 \rangle$. The even intertwining operator $J = J_1 \otimes J_2$ satisfies $J^2 = -1$. The restrictions $\rho_{\pm i}$ of ρ onto the eigensubspaces V_i and V_{-i} corresponding to eigenvalues $\pm i$ of J are irreducible and are transformed from each other by the change of parity functor.

It is easy to verify that given Q -bases

$$(v_1, \dots, v_n, J_1 v_1, \dots, J_1 v_n) \text{ and } (w_1, \dots, w_n, J_2 w_1, \dots, J_2 w_n)$$

of the spaces V_1 and V_2 respectively, the basis of $V_{\pm i}$ is the form

$$iv_k \otimes w_l \pm J_1 v_k \otimes J_2 w_l, v_r \otimes J_2 w_s + i J_1 v_r \otimes w_s; \quad k, r = 1, \dots, n; \quad l, s = 1, \dots, m.$$

Clearly, $\dim V_{\pm i} = (nm, nm)$ and the homogeneous subspaces of $V_{\pm i}$ may be identified with the tensor products

$$(V_{\pm i})_{\bar{0}} = (V_1)_{\bar{0}} \otimes (V_2)_{\bar{0}}, \quad (V_{\pm i})_{\bar{1}} = (V_1)_{\bar{1}} \otimes (V_2)_{\bar{0}}.$$

Let $(A, B) \in \mathfrak{g}_1$ and $(C, D) \in \mathfrak{g}_2$, then

$$\rho_{\pm i}((A, B)) = (A \otimes 1, B \otimes 1), \quad \rho_{\pm i}((C, D)) = (1 \otimes C, 1 \otimes (\pm i D)) \quad (1.3)$$

1.2.1. Theorem . *The image $\rho_{\pm i}(\mathfrak{g})$ is a maximal subalgebra of $\mathfrak{sl}(V_{\pm i})$.*

Proof. It is clear from (1.3), that the restriction of ρ_i to $\mathfrak{f}_1 = \mathfrak{q}(n_1) \oplus \mathfrak{q}(n_2)_{\bar{0}} \equiv \mathfrak{q}(n_1) \oplus \mathfrak{gl}(n_2)$ is the tensor product of the standard representations of \mathfrak{f}_1 in V_1 and $\mathfrak{q}(n_2)_{\bar{0}} = \mathfrak{gl}(n_2)$ in $(V_2)_{\bar{0}}$. As follows from the proved above, \mathfrak{f}_1 is a maximal subalgebra of the Lie superalgebra $\mathfrak{a}_1 = C(J_1 \otimes 1) = \left\{ \begin{pmatrix} A & B \\ B & A \end{pmatrix} \right\} \subset \mathfrak{gl}(nm|nm)$.

Similarly, $\rho_i|_{\mathfrak{f}_2}$ for $\mathfrak{f}_2 = \mathfrak{q}(n)_{\bar{0}} \oplus \mathfrak{g}_2$ is the tensor product of the standard representation of $\mathfrak{q}(n)_{\bar{0}} = \mathfrak{gl}(n)$ in $(V_1)_{\bar{0}}$ and $\mathfrak{q}(m)$ in V_2 . Therefore, \mathfrak{f}_2 is a maximal subalgebra in the Lie superalgebra

$$\mathfrak{a}_2 = C(1 \otimes J_2) = \left\{ \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \right\} \subset \mathfrak{gl}(nm|nm).$$

Since \mathfrak{a}_1 and \mathfrak{a}_2 generate the whole $\mathfrak{sl}(V_i)$, it follows that \mathfrak{g} is maximal in $\mathfrak{sl}(V_i)$. \square

Let us formulate this in another way.

1.2.2. Theorem . *The Lie subsuperalgebra $\mathfrak{g} = \mathfrak{q}(V_1 \otimes \mathcal{A}) \odot \mathfrak{q}(\mathcal{A} \otimes V_2)$ is maximal in $\mathfrak{sl}(V_1 \otimes \mathcal{A} \otimes V_2)$.*

1.3. Let $\dim V_1 = (r, s)$, $\mathfrak{n} = \mathfrak{gl}(V_1) \otimes \Lambda(n)$ and \mathfrak{g} the semidirect sum of the ideal \mathfrak{n} and the subalgebra $\mathfrak{vect}(0|n) = \mathfrak{k}$ with the natural action on the ideal. The Lie superalgebra \mathfrak{g} has a natural faithful representation ρ in $V = V_1 \otimes \Lambda(n)$ defined by the formulas

$$\begin{aligned} \rho(A \otimes \varphi)(v \otimes \psi) &= (-1)^{p(D)p(v)} A v \otimes \varphi \psi \\ \rho(D)(v \otimes \psi) &= -(-1)^{p(D)p(v)} v \otimes D(\psi) \end{aligned} \text{ for } A \otimes \varphi \in \mathfrak{n}, D \in \mathfrak{k}, v \otimes \psi \in V.$$

In the sequel, we will always identify elements of \mathfrak{g} with their images under ρ . Therefore, we will consider \mathfrak{g} embedded in $\mathfrak{gl}(V)$ which coincides, as a set, with $\text{Mat}(V) \cong \text{Mat}(V_1) \otimes \text{Mat}(\Lambda(n))$.

Finally, let us realize the associative superalgebra $\text{Mat}(\Lambda(n))$ as the associative superalgebra $\text{Diff}(n) := \text{Diff}(0|n)$ of differential operators acting on the superspace of functions $\Lambda(n)$ on $\mathcal{C}^{0,n}$.

Set $\text{grDiff}(n) = \oplus_{0 \leq i \leq n} \text{Diff}_i(n)$, where $\text{Diff}_i(n)$ is the superspace of homogeneous (w.r.t. order) order i differential operators. Observe that $\text{Diff}_0(n) = \Lambda(n)$, and $(\text{Diff}_1(n))_L \cong \mathfrak{vect}(0|n)$. In such a realization the bracket in $\mathfrak{gl}(V)$ is of the form

$$[A_1 \otimes D_1, A_2 \otimes D_2] = (-1)^{p(D_1)p(A_2)}([A_1, A_2] \otimes D_1 D_2 + (-1)^{p(A_1)p(A_2)} A_2 A_1 \otimes [D_1, D_2]). \quad (1.4)$$

Notice that the subset of matrices of the form $\begin{pmatrix} \alpha & \beta \\ 0 & \alpha \end{pmatrix}$ in $\text{Mat}(1|1)$ is, on the one hand, an associative subalgebra of $\text{Mat}(1|1)$ and, on the other hand, is a commutative ideal in $\mathfrak{gl}(1|1)$. Therefore, thanks to (1.4)

$$\tilde{\mathfrak{g}} = \mathfrak{gl}(1|1) \otimes \Lambda(n) + \left\{ \begin{pmatrix} \alpha & \beta \\ 0 & \alpha \end{pmatrix} \right\} \otimes \text{Diff}_1(n)$$

is a Lie superalgebra. Clearly, $\mathfrak{gl}(1|1) = \Lambda(1) \oplus \mathfrak{vect}(0|1)$ and

$$\tilde{\mathfrak{g}} = [\Lambda(1) \oplus \mathfrak{vect}(0|1)] \otimes \Lambda(n) \oplus \Lambda(1) \otimes \text{Diff}_1(n) \cong \Lambda(n+1) \oplus \mathfrak{vect}(0|n+1).$$

1.3.1. Theorem . 1) In all cases except a) $\dim V_1 = (1, 1)$ or b) $n = 1$ and $\dim V_1 = (1, 0)$ or $(0, 1)$ or (r, s) with $r \neq s$ the Lie superalgebra $\mathfrak{g} = \mathfrak{gl}(V_1) \otimes \Lambda(n) \oplus \mathfrak{vect}(0|n)$ is a maximal Lie subsuperalgebra of $\mathfrak{sl}(V)$, where $V = V_1 \otimes \Lambda(n)$.

2) If $\dim V_1 = (1, 1)$, then $\mathfrak{g} \subset \tilde{\mathfrak{g}}$ and $\tilde{\mathfrak{g}}$ is a maximal Lie subsuperalgebra in $\mathfrak{sl}(V)$.

3) If $n = 1$ and $\dim V_1 = (1, 0)$ or $(0, 1)$, then $\mathfrak{g} = \mathfrak{gl}(V)$.

4) If $n = 1$ and $\dim V_1 = (r, s)$, where $r \neq s$ and $r + s > 1$, then \mathfrak{g} is maximal Lie subsuperalgebra of $\mathfrak{gl}(V)$. The Lie superalgebra $\mathfrak{sg} := \mathfrak{gl}(V_1) \otimes \Lambda(1) \oplus \langle \partial \rangle$ is maximal in $\mathfrak{sl}(V)$.

Remark . If $\dim V_1 = (1, 0), (0, 1)$ or $(1, 1)$, then \mathfrak{g} (or $\tilde{\mathfrak{g}}$) is a Lie superalgebra of a general form (not semisimple or likewise). In other cases \mathfrak{g} is semisimple.

Set $\widetilde{\text{Diff}}_n(n) = \{\varphi \partial_1 \dots \partial_n \mid \deg \varphi < n \text{ in the standard grading of } \Lambda(n)\}$.

1.3.2. Lemma . 1) $\mathfrak{sl}(\Lambda(n)) \cong \oplus_{0 \leq i \leq n-1} \text{Diff}_i(n) \oplus \widetilde{\text{Diff}}_n(n)$ (as superspaces).

2) The representation of $\mathfrak{vect}(0|n)$ in $\text{Diff}_i(n)$ is irreducible for $n > 2$ if $i < n$ and contains the irreducible subsuperspace $\widetilde{\text{Diff}}_n(n)$ of codimension $(1, 0)$ if $i = n$.

Proof. 1) Let $F = \varphi \partial_{i_1} \dots \partial_{i_k} \in \text{Diff}(n)$ and $\deg \varphi = l$. Then $\deg F = l - k$. Hence, if $k \neq l$, then F is a nilpotent operator and $\text{str} F = 0$. If $k = l$, then $F(\Lambda(n)) \subset \varphi \Lambda(n)$. If, moreover, $\partial_{i_1} \dots \partial_{i_k} \varphi = 0$, then

$$F^2(\Lambda(n)) \subset F(\varphi \Lambda(n)) \subset F(\varphi) \Lambda(n) + \varphi \sum_{1 \leq l \leq k-1} \sum_{(j_2, \dots, j_l) \subset (i_1, \dots, i_k)} (\partial_{j_1} \dots \partial_{j_l} \varphi) \Lambda(n) = 0.$$

Thus, $\text{str} F$ can be nonzero only if $\varphi = \xi_{i_2} \dots \xi_{i_k}$.

Let us calculate $\text{str} F$ for $F = \xi_1 \dots \xi_k \partial_1 \dots \partial_k$ and $k < n$. We have

$$F| \oplus_{0 \leq i \leq k-1} \Lambda^i(n) = 0; \quad \text{tr} F| \Lambda^{k+i}(n) = \binom{n-k}{i} \text{ for } 0 \leq i \leq n-k$$

implying $(-1)^k \text{str} F = \sum_{0 \leq i \leq n-k} (-1)^i \binom{n-k}{i} = 0$.

2) Follows easily from [BL]. □

Proof of Theorem 1.3.1. Cases 2) and 3) had been actually considered even before we have formulated the theorem. In 4) it clearly suffices to prove the maximality of \mathfrak{g} in $\mathfrak{gl}(V)$. We will prove this together with 1).

Let \mathfrak{h} be an intermediate Lie subsuperalgebra: $\mathfrak{g} \subset \mathfrak{h} \subset \mathfrak{gl}(V)$. Consider two cases:

1°. Let $\dim V_1 = (1, 0)$ or $(0, 1)$, $n > 1$, i.e., $V = \Lambda(n)$ or $\pi(\Lambda(n))$; $\mathfrak{g} = \Lambda(n) \oplus \mathfrak{vect}(0|n)$ and $\mathfrak{gl}(V) \cong$

$\text{Diff}(n)_L$. Let $\alpha = (i_1, \dots, i_k)$ be a multiindex, $D_\alpha = \partial_{i_1} \dots \partial_{i_k}$, $D_\alpha^i = \begin{cases} \partial_{i_1} \dots \partial_{i_k} & \text{if } i \in \alpha \\ 0 & \text{otherwise.} \end{cases}$ Then

$$[\sum \varphi_\alpha D_\alpha, \xi_i] = \sum (-1)^{p(D_\alpha)} \varphi_\alpha D_\alpha^i \quad (1.5)$$

a) It follows from (1.5) that by commuting with functions ξ_1, \dots, ξ_n we can obtain an arbitrary 2nd order differential operator from any differential operator of order ≥ 2 . It follows that if $\mathfrak{h} \neq \mathfrak{g}$, then \mathfrak{h} contains at least one 2nd order operator.

b) The representation of $\mathfrak{k} = \mathfrak{vect}(0|n) \subset \mathfrak{g}$ in $\text{Diff}_2(n)$ is irreducible if $n > 2$ by Lemma 1.3.2. Therefore, if $\mathfrak{h} \neq \mathfrak{g}$, then \mathfrak{h} contains $\text{Diff}_2(n)$ if $n > 2$ and contains $\widetilde{\text{Diff}}_2(2)$ if $n = 2$.

c) Finally, it is not difficult to see that $\text{Diff}_2(n)$ generates $\sum_{2 \leq i \leq n-1} \text{Diff}_i(n) + \widetilde{\text{Diff}}_n(n)$ and, therefore, if $\mathfrak{h} \neq \mathfrak{g}$, then $\mathfrak{h} = \mathfrak{sl}(V)$.

2° Let $\dim V_1 = (r, s)$ and $r + s \geq 2$. a) It follows from (1.4) that if $A \in \mathfrak{gl}(V_1)$, $\varphi \in \Lambda(n)$, $D \in \text{Diff}(n)$, then

$$[A \otimes \varphi, 1 \otimes D] = A \otimes [\varphi, D].$$

Since any operator of the form $A \otimes \varphi$ belongs to the considered subalgebra \mathfrak{g} , it follows that if $\mathfrak{h} \neq \mathfrak{g}$, then \mathfrak{h} contains at least one operator with the first component different from 1.

b) Formulas (1.4) and (1.5) imply that commuting with operators of the form $1 \otimes \xi_i$ enables one to obtain from any operator of order ≥ 2 a 1st order operator. Taking a) into account we conclude that \mathfrak{h} must contain at least one 1st order operator which does not belong to \mathfrak{g} .

c) The Lie superalgebra \mathfrak{g} contains as a Lie subalgebra the direct sum of Lie superalgebras

$$\begin{array}{ll} \mathfrak{gl}(V_1) \otimes 1 \oplus 1 \otimes \mathfrak{vect}(0|n) & \text{if } n \neq 1 \\ \mathfrak{gl}(V_1) \otimes 1 \oplus 1 \otimes \langle \partial \rangle & \text{if } n = 1 \end{array}$$

and in the space $\mathfrak{gl}(V_1) \otimes \mathfrak{vect}(0|n)$ the tensor product of the adjoint representations of Lie superalgebras $\mathfrak{gl}(V_1)$ and $\mathfrak{vect}(0|n)$ acts. Therefore, \mathfrak{h} contains the set $U_1 \otimes U_2$, where U_1 is ideal of $\mathfrak{gl}(V_1)$ different from $\mathbb{C} \cdot 1$ and U_2 is an ideal of $\mathfrak{vect}(0|n)$. Since we have excluded the case $\dim V_1 = (1, 1)$, this means that $U_1 \supset \mathfrak{sl}(V_1)$ and $U_2 = \mathfrak{vect}(0|n)$ for $n \neq 1$ and $U_2 \supset \mathbb{C}\partial_1$ for $n = 1$ (due to simplicity of $\mathfrak{vect}(0|n)$ for $n \neq 1$).

Let $A \in \mathfrak{gl}(V_1)$, $B \in \mathfrak{sl}(V_1)$. Then taking the bracket of $A \otimes \xi$ and $B \otimes \partial$ according to (1.4):

$$[A \otimes \xi, B \otimes \partial] = (-1)^{p(B)}([A, B] \otimes \xi \partial + (-1)^{p(A)p(B)} BA \otimes 1)$$

we see that U_2 coincides with $\mathfrak{vect}(0|1)$ for $n = 1$, too. Therefore, $\mathfrak{h} \supset \mathfrak{sl}(V_1) \otimes \mathfrak{vect}(0|n)$.

d) Let $D, D' \in \mathfrak{vect}(0|n)$. Set

$$A_0 = \begin{cases} \text{diag}(1, -1, 0, \dots, 0) \in \mathfrak{sl}(V_1) & \text{for } r > 1 \\ \text{diag}(0, 1, -1, 0, \dots, 0) \in \mathfrak{sl}(V_1) & \text{for } r = 1. \end{cases}$$

Then $[A_0 \otimes D, A_0 \otimes D'] = A_0^2 \otimes [D, D']$, and $\text{str} A_0^2 \neq 0$. This implies

$$\mathfrak{h} \supset \begin{cases} \mathfrak{gl}(V_1) \otimes \mathfrak{vect}(0|n) & \text{if } n > 1 \\ \mathfrak{gl}(V_1) \otimes \langle \partial \rangle + \mathfrak{sl}(V_1) \otimes \langle \xi \partial \rangle & \text{if } n = 1. \end{cases}$$

e) Further, let us carry out the induction on the order of the differential operator. Let $\mathfrak{h} \supset \sum_{0 \leq i \leq k} \mathfrak{gl}(V_1) \otimes \text{Diff}_i(n)$. By bracketing $\mathfrak{gl}(V_1) \otimes \mathfrak{vect}(0|n)$ according to (1.4) we get $\mathfrak{h} \supset \mathfrak{sl}(V_1) \otimes \text{Diff}_{k+1}(n)$. By bracketing $A_0 \otimes D_1$ with $A_0 \otimes D_{k+1}$ for $D_1 \in \mathfrak{vect}(0|n)$ and $D_{k+1} \in \text{Diff}_{k+1}(n)$ we get an operator $A_0^2 \otimes [D_1, D_{k+1}] \in \mathfrak{gl}(V_1) \otimes \text{Diff}_{k+1}(n)$ that does not belong to $\mathfrak{sl}(V_1) \otimes \text{Diff}_{k+1}(n)$.

Let us consider how $\mathfrak{gl}(V_1) \otimes 1$ acts in the space $\mathfrak{gl}(V_1) \otimes \text{Diff}_{k+1}(n)$. We see that

$$\begin{aligned} \mathfrak{h} &\supset \mathfrak{sl}(V_1) \otimes \text{Diff}_{k+1}(n) + \mathfrak{gl}(V_1) \otimes [\mathfrak{vect}(0|n), \text{Diff}_{k+1}(n)] = \\ &= \begin{cases} \mathfrak{gl}(V_1) \otimes \text{Diff}_{k+1}(n) & \text{if } k+1 \neq 1 \\ \mathfrak{gl}(V_1) \otimes \widetilde{\text{Diff}}_n(n) + \mathfrak{sl}(V_1) \otimes \text{Span}(\xi_1 \dots \xi_n \partial_1 \dots \partial_n) & \text{if } k+1 = n. \quad \square \end{cases} \end{aligned}$$

1.4. Let $\mathfrak{hei}(0|2n)$ be generated by odd $\xi_1, \dots, \xi_n, \eta_1, \dots, \eta_n$ and an even z , see sec. 0.2.

The nonzero irreducible finite dimensional representation of $\mathfrak{hei}(0|2n)$ that maps z to the operator of multiplication by 1 is realized in the superspace $\Lambda(n)$ of functions in n odd variables $\xi = (\xi_1, \dots, \xi_n)$. The normalizer of $\mathfrak{hei}(0|2n)$ in $\mathfrak{gl}(\Lambda(n))$ is $\mathfrak{g} = \mathfrak{hei}(0|2n) \oplus \mathfrak{o}(2n)$; it acts in the spinor representation of $\mathfrak{o}(2n)$, or, in terms of differential operators: $\mathfrak{g} = \text{Span}(1, \xi, \partial, \xi \partial, \partial \partial, \xi \xi)$.

Let $n = 3$. Then \mathfrak{g} is contained in $\tilde{\mathfrak{g}} = (\mathfrak{hei}(0|6) \in V) \in \mathfrak{o}(6)$ (sum as $\mathfrak{o}(6)$ -modules), where the highest weight of the $\mathfrak{o}(6) \simeq \mathfrak{sl}(4)$ -module V is $(2, 0, 0)$, i.e., $\tilde{\mathfrak{g}}$ is isomorphic to the nontrivial central extension \mathfrak{as} of $\mathfrak{spe}(4)$. (Observe, that $\mathfrak{sl}(4)$ -module $\mathfrak{hei}(0|6)$ is the direct sum of the trivial module and the exterior square of the dual to the standard 4-dimensional $\mathfrak{sl}(4)$ -module.)

Theorem . \mathfrak{g} is a maximal subalgebra in $\mathfrak{sl}(\Lambda(n))$ for $n \geq 2$ and $n \neq 3$. If $n = 3$, then $\mathfrak{g} \subset \tilde{\mathfrak{g}} \cong \mathfrak{as}$, and \mathfrak{as} is maximal in $\mathfrak{sl}(\Lambda(n))$.

Proof. Let us realize again $\mathfrak{sl}(\Lambda(n))$ by differential operators

$$\mathfrak{sl}(\Lambda(n)) = \oplus_{i \leq n-1} \text{Diff}_i(n) \oplus \widetilde{\text{Diff}}_n(n). \quad (1.6)$$

Let \mathfrak{h} be a Lie subsuperalgebra in $\mathfrak{sl}(\Lambda(n))$ containing \mathfrak{g} and strictly greater than it.

1° First, let us show that \mathfrak{h} contains an element $h \notin \mathfrak{g}$ whose highest term with respect to the grading (1.6) is of degree 1. Let $h \in \mathfrak{h}$ and $h \notin \mathfrak{g}$. Note that

$$[\varphi \partial_{i_1} \dots \partial_{i_k}, \xi_i] = \begin{cases} 0 & \text{for } i \notin (i_1, \dots, i_k) \\ \pm \varphi \partial_{i_1} \dots \hat{\partial}_i \dots \partial_{i_k} & \text{for } i \in (i_1, \dots, i_k) \end{cases}$$

Since $\xi_i \in \mathfrak{g}$ for all $i = 1, \dots, n$, it follows that if $\deg h \geq 2$, then commuting with ξ_i we may obtain an element \tilde{h} of degree 1 from h . Now, since $\xi_i \xi_j \in \mathfrak{g}$ for all $(i, j) \in (1, \dots, n)$ and

$$\begin{aligned} &[\varphi \partial_{i_1} \dots \partial_{i_k}, \xi_i \xi_j] \\ &= \begin{cases} 0 & \text{for } i, j \notin (i_1, \dots, i_k) \\ \pm \xi_j \varphi \partial_{i_1} \dots \hat{\partial}_i \dots \partial_{i_k} & \text{for } i \in (i_1, \dots, i_k), j \notin (i_1, \dots, i_k) \\ \pm \xi_i \varphi \partial_{i_1} \dots \hat{\partial}_j \dots \partial_{i_k} & \text{for } i \notin (i_1, \dots, i_k), j \in (i_1, \dots, i_k) \\ \pm \xi_i \varphi \partial_{i_1} \dots \hat{\partial}_j \dots \partial_{i_k} & \text{for } i, j \in (i_1, \dots, i_k), \\ \pm \xi_j \varphi \partial_{i_1} \dots \hat{\partial}_i \dots \partial_{i_k} \pm \varphi \partial_{i_1} \dots \hat{\partial}_i \dots \hat{\partial}_j \dots \partial_{i_k} & \text{for } i, j \in (i_1, \dots, i_k), \end{cases} \end{aligned}$$

we may assume that $\tilde{h} \notin \mathfrak{g}$. If $\deg h = 0$, i.e., $h \in \Lambda(n)$, it suffices to consider brackets of the form

$$[\partial_i \partial_j, h] = \pm \partial_i(h) \partial_j \pm \partial_j(h) \partial_i \pm \partial_i \partial_j(h)$$

and notice that since $h \notin \mathfrak{g}$, then $\deg h \geq 3$ with respect to the grading in $\Lambda(n)$.

2°. Let $n = 2$. We have got that \mathfrak{h} contains an element of the form $\xi_1 \cdot \xi_2 (\alpha \partial_1 + \beta \partial_2)$ for some $(\alpha, \beta) \neq (0, 0)$. Since $\mathfrak{gl}(2) = \langle \xi_i \partial_j \rangle$ acts irreducibly in the space $\langle \xi_1 \xi_2 \partial_1, \xi_1 \xi_2 \partial_2 \rangle$, then $\mathfrak{h} \supset \Lambda(2) \oplus \mathfrak{vect}(0|2)$ and by Theorem 1.3.1 $\mathfrak{h} = \mathfrak{sl}(\Lambda(2))$.

3°. Let $n \geq 3$. By 1°, we may assume that \mathfrak{g} contains a set of elements whose highest terms run over $\mathfrak{svect}(0|n) \subset \mathfrak{vect}(0|n)$. Acting by them on functions contained in \mathfrak{g} we see that \mathfrak{h} contains $T^0(n) \oplus \mathfrak{svect}(0|n)$, where $T^0(n) \cong \Lambda(n) / \langle \xi_1 \dots \xi_n \rangle$ is the superspace of functions with integral 0.

For $n = 3$ it is easy to see that $\mathfrak{g} \oplus T^0(3) \oplus \mathfrak{svect}(0|3)$ generates $\tilde{\mathfrak{g}}$.

If $n > 3$, then the relation

$$[\partial_1 \partial_2, \xi_1 \xi_2 \xi_3] = (\xi_1 \partial_1 + \xi_2 \partial_2 - 1) \xi_3 \notin \mathfrak{svect}(0|n)$$

implies that the sets $T^0(n)$ and $\langle \partial_i \partial_j, 1 \leq j \leq n \rangle$ generate $\mathfrak{vect}(0|n)$; hence, $\mathfrak{h} \supset \Lambda(n) \oplus \mathfrak{vect}(0|n)$ and, by Theorem 1.3.1, $\mathfrak{h} = \mathfrak{sl}(\Lambda(n))$.

The maximality of $\tilde{\mathfrak{g}}$ for $n = 3$ is obvious. \square

§2. PROOF OF MAIN THEOREM

In this section \mathfrak{g} is an irreducible matrix Lie superalgebra, ρ its standard representation in superspace W . Let \mathfrak{k} be an ideal of \mathfrak{g} . We will assume that $\dim W > 1$ (because the case $\dim W = 1$ is trivial). Our proof of Main Theorem is largely based on the following Theorem 2.0 — the superization of a statement well-known to the reader from Dixmier's book [Di].

2.0. Theorem . *Let \mathfrak{g} be an irreducible matrix Lie superalgebra, ρ its standard representation (in particular, ρ is faithful). Let \mathfrak{k} be an ideal of \mathfrak{g} and $\dim \mathfrak{k} > 1$. Then 3 cases are possible:*

- A) $\rho|_{\mathfrak{k}}$ is irreducible;
- B) $\rho|_{\mathfrak{k}}$ is a multiple of an irreducible \mathfrak{k} -module τ and the multiplicity of τ is > 1 ;
- C) there exists a subalgebra $\mathfrak{h} \subset \mathfrak{g}$ such that $\mathfrak{k} \subset \mathfrak{h}$ and $\rho \equiv \text{ind}_{\mathfrak{h}}^{\mathfrak{g}} \sigma$ for an irreducible \mathfrak{k} -module σ .

We will say that the representation ρ is of type A, B, or C with respect to the ideal \mathfrak{k} if the corresponding case holds. Lemmas 2.4, 2.1 and 2.2 deal with types A, B and C, respectively.

Proof largely follows the lines of [Di] with one novel case: irreducible modules of Q -type might occur. Fortunately, their treatment is rather straightforward and I would rather save paper by omitting the verification.

2.1. Lemma . B) 1) *If ρ is of type B with respect to the ideal \mathfrak{k} , then either $\mathfrak{g} \subset \mathfrak{gl}(V_1) \odot \mathfrak{gl}(V_2)$ or $\mathfrak{g} \subset \mathfrak{q}(V_1) \odot \mathfrak{q}(V_2)$ for some V_1 and V_2 .*

2) *If, moreover, $\mathfrak{g} \subset \mathfrak{q}(V)$, then $\mathfrak{g} \subset \mathfrak{q}(V_1) \odot \mathfrak{gl}(V_2)$ for some V_1, V_2 ;*

3) *If ω is a nondegenerate 2-form on V and $\mathfrak{g} \subset \mathfrak{aut}(\omega)$, then $\mathfrak{g} \subset \mathfrak{aut}(\omega_1) \oplus \mathfrak{aut}(\omega_2)$, where $\omega = \omega_1 \otimes \omega_2$*

Proof. 1) Let V be the superspace of representation τ . Then $V = V_1 \otimes U$ for some U and

$$\rho(h)(v \otimes u) = \tau(h)v \otimes u \text{ for any } h \in \mathfrak{k}, v \in V_1, u \in U$$

i.e.,

$$\text{the operators } \rho(h) \text{ are of the form } \tau(h) \otimes 1. \quad (2.1)$$

On the other hand, \mathfrak{g} is the stabilizer of τ . So for any $g \in \mathfrak{g}$ there exists $s(g) \in \text{End} V_1$ such that $\tau([g, h]) = [s(g), \tau(h)]$ for any $h \in \mathfrak{k}$.

Let us consider the operator $T(g) = \rho(g) - s(g) \otimes 1$ in V . For $h \in \mathfrak{k}$ we have

$$[T(g), \rho(h)] = [\rho(g) - s(g) \otimes 1, \rho(h)] = \rho([g, h]) - [s(g), \tau(h)] \otimes 1 = 0,$$

i.e., $T(g)$ commutes with all the operators of the form $\rho(h)$, $h \in \mathfrak{k}$ and since τ is irreducible, $T(g)$ is of the form

$$T(g) = \begin{cases} 1 \otimes A(g) & \text{if } \tau \text{ is of } G\text{-type} \\ 1 \otimes A(g) + J \otimes B(g) & \text{if } \tau \text{ is of } Q\text{-type} \end{cases},$$

where J is an odd generator of $C(\tau)$, the centralizer of τ .

Therefore, $\rho(g)$ is either of the form

$$\rho(g) = s(g) \otimes 1 + 1 \otimes A(g), \text{ if } \tau \text{ is of } G\text{-type} \quad (2.2)$$

or

$$\rho(g) = s(g) \otimes 1 + 1 \otimes A(g) + J \otimes B(g) \text{ if } \tau \text{ is of } Q\text{-type.} \quad (2.3)$$

Formulas (2.1) – (2.3) show that if τ is of G -type, then \mathfrak{g} is contained in $\mathfrak{gl}(V_1) \odot \mathfrak{gl}(V_2) \subset \mathfrak{gl}(V)$, where $V_2 = U$, and if τ is of Q -type, then \mathfrak{g} is contained in $\mathfrak{q}(V_1) \odot \mathfrak{q}(V_2) \subset \mathfrak{gl}(V)$, where $V_2 = \text{Span}(1, J) \otimes U$.

2) To prove this heading it suffices to notice that if τ is of type G and the operators $T(g) = 1 \otimes A(g)$ determine a G -irreducible action on U , then ρ is also of type G . Therefore, either $\mathfrak{g} \subset \mathfrak{q}(V_1) \odot \mathfrak{gl}(V_2)$, where $V_2 = U$, or $\mathfrak{g} \subset \mathfrak{gl}(V_1) \odot \mathfrak{q}(U)$.

3) To prove this heading, observe that if we fix $u_1, u_2 \in U$, then on V there arises a family of \mathfrak{k} -invariant 2-forms $\omega_1^{u_1, u_2}$. Since τ is irreducible, each form $\omega_1^{u_1, u_2}$ is either nondegenerate or identically zero. Since ω is nondegenerate, there is a nondegenerate form $\omega_1 = \omega_1^{u_1, u_2}$ among the lot. In this case the whole space of forms is 1-dimensional and τ is a G -irreducible representation. For Lie algebras this is a well-known fact; for Lie superalgebras we prove it in Appendix 2.

Therefore, $\omega(v_1 \otimes u_1, v_2 \otimes u_2) = \omega_1(v_1, v_2) \cdot \omega_2(u_1, u_2)$, where ω_2 is the nondegenerate bilinear form on U . The action of an element $\rho(g)$ of the form (2.2) on ω gives:

$$\begin{aligned} 0 &= (s(g) \otimes 1 + 1 \otimes A(g))\omega(v_1 \otimes u_1, v_2 \otimes u_2) = \\ &= (\omega_1(s(g)(v_1), v_2) + (-1)^{p(g)p(v_1)}\omega_1(v_1, s(g)(v_2)))\omega_2(u_1, u_2) = \\ &= \omega_1(v_1, v_2) (\omega_2(A(g)(u_1), u_2) + (-1)^{p(g)p(u_1)}\omega_2(u_1, A(g)(u_2))). \end{aligned}$$

After simplification we get:

$$s(g)(\omega_1) = c(g)\omega_1; \quad A(g)\omega_2 = -c(g)\omega_2,$$

where $c(g) \in \mathbb{C}$.

Therefore,

$$\rho(g) = s(g) \otimes 1 + 1 \otimes A(g) = (s(g) - \frac{1}{2}c(g) \cdot \text{id}) \otimes 1 + 1 \otimes (A(g) + \frac{1}{2}c(g) \cdot \text{id}),$$

where $s(g) - \frac{1}{2}c(g) \cdot \text{id} \in \mathfrak{aut}(\omega_1)$ and $\text{id} \otimes (A(g) + \frac{1}{2}c(g) \cdot \text{id}) \in \mathfrak{aut}(\omega_2)$. \square

2.2. Lemma . C) 1) If ρ is of type C with respect to the ideal \mathfrak{k} , then $\mathfrak{g} \subset \mathfrak{gl}(V) \otimes \Lambda(n) \oplus \mathfrak{vect}(0|n)$ for some V and n .

2) If, moreover, $\mathfrak{g} \subset \mathfrak{q}(n)$, then either $\mathfrak{g} \subset \mathfrak{q}(V) \otimes \Lambda(n) \oplus \mathfrak{vect}(0|n)$ or $n = 1$ and $\mathfrak{g} \subset \mathfrak{gl}(V) \otimes \text{id}_{\Lambda(1)} \oplus \mathbb{C}(\xi + \partial\xi)$;

3) If ρ is of type C with respect to the ideal \mathfrak{k} and $\mathfrak{g} \subset \mathfrak{aut}(\omega)$, then $\mathfrak{g} \subset \mathfrak{aut}(\omega_1) \otimes \Lambda(n) \oplus T^{1/2}(\mathfrak{vect}(0|n))$, where $\omega = \omega_1 \otimes \omega_2$ and the representation $T^{1/2}$ of $\mathfrak{vect}(0|n)$ (in the superspace of halfdensities) is given by the formula $T^{1/2}(D) = 1 \otimes (D + \frac{1}{2}\text{div}D)$ and the form ω_2 on $\Lambda(\xi)$ is given by the formula

$$\omega_2(f\sqrt{\text{vol}(\xi)}, g\sqrt{\text{vol}(\xi)}) = \int fg \cdot \text{vol}(\xi).$$

Remark . 1) $\mathfrak{gl}(V) \otimes \text{id}_{\Lambda(1)} \oplus \mathbb{C}(\xi + \partial\xi) \cong \mathfrak{gl}(V) \odot \mathfrak{q}(1)$.

2) $\mathfrak{aut}(\omega_1) \otimes 1 \oplus T^{1/2}(\mathfrak{vect}(0|n)) \subset \mathfrak{aut}(\omega_1) \odot \mathfrak{aut}(\omega_2)$

Lemma C follows from the definition of the induced representaiton.

2.3. To complete the proof of Main Theorem it suffices to consider the case when ρ possesses the following property: for any ideal $\mathfrak{k} \subset \mathfrak{g}$ either ρ is of type A with respect to \mathfrak{k} or $\rho|_{\mathfrak{k}}$ is the multiple of a character.

Let \mathfrak{r} be the radical of the matrix Lie superalgebras \mathfrak{g} .

Lemma . If \mathfrak{r} is commutative and ρ belongs to type A or B with respect to \mathfrak{r} , then $\dim \mathfrak{r} = 0$ or 1.

Proof follows from the fact that any irreducible representation of a commutative Lie superalgebra is 1-dimensional (up to the change of parity) and the faithfulness of ρ . \square

2.4. In the case when \mathfrak{r} is not commutative, consider the derived series of \mathfrak{r} :

$$\mathfrak{r} \supset \mathfrak{r}_1 \supset \cdots \supset \mathfrak{r}_k \supset \mathfrak{r}_{k+1} = 0,$$

where $\mathfrak{r}_{i+1} = [\mathfrak{r}_i, \mathfrak{r}_i]$. Clearly, each \mathfrak{r}_i is an ideal in \mathfrak{g} and the last ideal, \mathfrak{r}_k is commutative.

Lemma . A) 1) If ρ is of type A with respect to \mathfrak{r}_{k-1} and $\rho|_{\mathfrak{r}_k}$ is scalar, then either $\mathfrak{r}_{k-1} \cong \mathfrak{hei}(0|2n)$ or $\mathfrak{r}_{k-1} \cong \mathfrak{hei}(0|2n-1)$ and $V \cong \Lambda(n)$ or $\Pi(\Lambda(n))$ and $\mathfrak{g} \subset \mathfrak{hei}(0|2n) \oplus \mathfrak{o}(2n)$.

2) If additionally $\mathfrak{g} \subset \mathfrak{q}(V)$, then either $\mathfrak{r}_{k-1} \cong \mathfrak{hei}(0|2n-1)$ or $\mathfrak{g} \subset \mathfrak{hei}(0|2n-1) \oplus \mathfrak{o}(2n-2) \cong (\mathfrak{hei}(0|2n-2) \oplus \mathfrak{o}(2n-2)) \odot \mathfrak{q}(1)$.

3) In this case \mathfrak{g} does not preserve any nondegenerate 2-form on V .

Proof. Headings 2 and 3 follow from heading 1. Let us prove it.

We see that if the restriction $\rho|_{\mathfrak{r}_i}$ is irreducible for all i , then

- 1) $\dim \mathfrak{r}_k = 1$ because ρ is faithful;
- 2) $\mathfrak{r}_{k-1} = \mathfrak{hei}(0|m)$ for any m ;
- 3) $\rho|_{\mathfrak{r}_{k-1}}$ is irreducible and faithful, so it can be realized in the superspace of functions, $\Lambda(2n)$, or in $\Pi(\Lambda(2n))$, where $n = [\frac{m}{2}] + 1$;
- 4) \mathfrak{g} is contained in the normalizer of $\mathfrak{hei}(0|2n)$ in its spinor representation, i.e., $\mathfrak{g} \subset \mathfrak{hei}(0|2n) \in \mathfrak{o}(2n)$. \square

2.5. Lemma . *Let $\dim \mathfrak{r} \leq 1$, i.e., \mathfrak{g} is either semisimple or a nontrivial central extension of a semisimple Lie superalgebra but NOT an almost simple or a central extension of an almost simple Lie superalgebra. Then we can always choose an ideal \mathfrak{k} such that ρ is of type B or C with respect to \mathfrak{k} .*

We prove this Lemma in Appendix 1.

Lemmas 2.1 (Lemma B), 2.2 (Lemma C), 2.3, 2.4 (Lemma A) and Lemma 2.5 put together prove Main Theorem. \square

§3. IRREDUCIBLE NON-SIMPLE MAXIMAL MATRIX SUBALGEBRAS OF $\mathfrak{q}(n)$

3.1. Let $\dim W = (n, n)$, $n \geq 2$ and $J = \begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix}$ (so, $J^2 = -1_V$ and $p(J) = \bar{1}$). Consider the centralizer $C(J) = \{A \in \mathfrak{gl}(V) : [A, J] = 0\}$ of J . Notice that on the set $C(J)$, as well as on the set of the all operators, two structures can be introduced: that of the associative superalgebra $Q(W)$ and that of the Lie superalgebra $\mathfrak{q}(W) = Q(W)_L$.

Now, let $V = V_1 \otimes V_2$ and $\dim V_1 = (n, n)$; let $\dim V_2 = (n_2, m_2)$ be arbitrary. Given an odd operator $J_1 \in \text{End}(V_1)$ such that $J_1^2 = -1_{V_1}$ (and, therefore, a superalgebra $C(J_1)$ being defined), then the operator $J = J_1 \otimes 1$ in V is defined such that $J^2 = -1_V$ and $p(J) = \bar{1}$. Observe that

$$Q\text{Mat}(V) \cong Q\text{Mat}(V_1) \odot \text{Mat}(V_2). \quad (3.1)$$

Consider the Lie superalgebra $\mathfrak{g} = \mathfrak{q}(V_1) \odot \mathfrak{gl}(V_2)$. Clearly, $\mathfrak{g} \subset \mathfrak{q}(V)$.

3.1.1. Lemma . *The Lie superalgebra \mathfrak{g} is irreducible on V if $n_2 \neq m_2$.*

Proof. The operators of the form $1 \otimes B$ for $B \in \mathfrak{gl}(V_2)$ commute with the operators of $\mathfrak{gl}(V)$ that are of the form $C \otimes 1$ for $C \in \mathfrak{gl}(V_1)$ and only with them, while operators of the form $A \otimes 1$ for $A \in \mathfrak{q}(V_1)$ commute with the operators of the form $1 \otimes C$ and $J_1 \otimes C$, where $C \in \mathfrak{gl}(V_2)$. Therefore, the algebra of intertwining operators of the considered representation of \mathfrak{g} is generated by two operators: 1 and $J_1 \otimes 1$. \square

3.1.2. Theorem . *The Lie subsuperalgebra $\mathfrak{g} = \mathfrak{q}(V_1) \odot \mathfrak{gl}(V_2)$ is maximal in $\mathfrak{q}(V_1 \otimes V_2)$.*

Proof. Formula (3.1) enables to actually repeat the proof of Theorem 1.1.

We must only notice that like $\mathfrak{gl}(W)$, the Lie superalgebra $\mathfrak{q}(W)$ contains two ideals only: that of “odd constants” and $\mathfrak{sq}(W) = \{X \in \mathfrak{q}(W) | \text{qtr} X = 0\}$ which is not endowed with the natural associative algebra structure. Besides, if $g \in \mathfrak{q}(V_1)$ but $g \notin \mathfrak{sq}(V_1)$, then $g \otimes 1 \notin \mathfrak{sq}(V)$. \square

3.2. Let V_1 be an (m, m) -dimensional linear superspace and $\mathfrak{q}(V_1) = \mathfrak{q}(m) = C(J)$ be the queer Lie superalgebra with the standard representation in V_1 . Let $V = V_1 \otimes \Lambda(n)$.

Theorem . *The Lie subsuperalgebra $\mathfrak{g} = \mathfrak{q}(V_1) \otimes \Lambda(n) \in \mathfrak{vect}(0|n)$ is maximal in $\mathfrak{q}(V) = C(J \otimes 1)$.*

Proof. We can repeat the proof of Theorem 1.3.1 by replacing $\mathfrak{gl}(V_1)$ with $\mathfrak{q}(V_1)$ and $\mathfrak{sl}(V_1)$ with $\mathfrak{sq}(V_1)$ provided in step d) we consider

$$[\Pi A_0 \otimes D, A_0 \otimes D'] = \Pi A_0^2 [D, D']$$

instead of $[A_0 \otimes D, A_0 \otimes D']$. \square

3.3. Adding together Theorems 3.1.2, 3.2 and heading 2° of Main Theorem we get the following statement.

Theorem . *If $\mathfrak{g} \subset \mathfrak{q}(n) = \mathfrak{q}(V)$ is a maximal in $\mathfrak{q}(n)$ irreducible Lie superalgebra, not almost simple nor a central extension of an almost simple, then either $\mathfrak{g} = \mathfrak{q}(V_1) \odot \mathfrak{gl}(V_2)$ for some V_1 and V_2 such that $V = V_1 \otimes V_2$ and $\dim V_2 > 1$ or*

$\mathfrak{g} = \mathfrak{q}(V_0) \otimes \Lambda(n) \in \mathfrak{vect}(0|n)$ for some V_0 and n such that $V = V_0 \otimes \Lambda(n)$.

§4. LIE SUPERALGEBRAS THAT PRESERVE A BILINEAR FORM

Let a nondegenerate homogeneous supersymmetric bilinear form ω be given in a superspace V . Sometimes, in a fixed basis, we will represent ω by its matrix F . Consider two objects associated with ω :

1) The space $\mathbf{aut}(\omega) = \mathbf{aut}(F)$ of operators preserving ω :

$$\mathbf{aut}(\omega) = \{A \in \mathfrak{gl}(V) \mid \omega(Ax, y) + (-1)^{p(A)(p(x)+p(\omega))} \omega(x, A^{st}y) = 0\}$$

or, in the matrix form,

$$\mathbf{aut}(F) = \{A \in \mathfrak{gl}(V) \mid AF + (-1)^{p(A)p(F)} FA^{st} = 0\}. \quad (4.1)$$

2) The space $\mathbf{symm}(\omega) = \mathbf{symm}(F)$ of operators symmetric with respect to ω :

$$\mathbf{symm}(\omega) = \{A \in \mathfrak{gl}(V) \mid \omega(Ax, y) = (-1)^{p(A)(p(x)+p(\omega))} \omega(x, A^{st}y)\}$$

or, in the matrix form,

$$\mathbf{symm}(F) = \{A \in \mathfrak{gl}(V) \mid AF = (-1)^{p(A)p(F)} FA^{st}\}. \quad (4.2)$$

Direct calculations yield the following statement:

4.1. Lemma . 1) *The space $\mathbf{aut}(\omega)$ is endowed with a natural Lie superalgebra structure while the superspace $\mathbf{symm}(\omega)$ is endowed with the structure of an $\mathbf{aut}(\omega)$ -module. Moreover, $[\mathbf{symm}(\omega), \mathbf{symm}(\omega)] \subset \mathbf{aut}(\omega)$.*

2) *Set $\{\cdot, \cdot\} : A, B \longrightarrow AB + (-1)^{p(A)p(B)} BA$. Then*

$$\begin{aligned} \{\mathbf{aut}(\omega), \mathbf{aut}(\omega)\}, \quad \{\mathbf{symm}(\omega), \mathbf{symm}(\omega)\} &\subset \mathbf{symm}(\omega); \quad \{\mathbf{aut}(\omega), \mathbf{symm}(\omega)\} \subset \mathbf{aut}(\omega); \\ \mathbf{aut}(\omega_1 \otimes \omega_2) &= \mathbf{aut}(\omega_1) \otimes \mathbf{symm}(\omega_2) + \mathbf{symm}(\omega_2) \otimes \mathbf{aut}(\omega_2); \\ \mathbf{symm}(\omega_1 \otimes \omega_2) &= \mathbf{aut}(\omega_1) \otimes \mathbf{aut}(\omega_2) + \mathbf{symm}(\omega_1) \otimes \mathbf{symm}(\omega_2). \end{aligned} \quad (4.3)$$

3) *Any $A \in \mathfrak{gl}(V)$ can be represented as the sum $A = A^a + A^s$, where $A^a \in \mathbf{aut}(\omega)$, $A^s \in \mathbf{symm}(\omega)$.*

Set

$$\mathbf{saut}(\omega) = \mathbf{aut}(\omega) \cap \mathfrak{sl}(V), \quad \mathbf{ssym}(\omega) = \mathbf{symm}(\omega) \cap \mathfrak{sl}(V).$$

Notice that if $p(\omega) = \bar{0}$, then $\mathbf{aut}(\omega) = \mathbf{saut}(\omega)$, and if $p(\omega) = \bar{1}$, then $\mathbf{symm}(\omega) = \mathbf{ssym}(\omega)$.

Theorem . *Let Lie superalgebras $\mathbf{saut}(\omega_1)$ and $\mathbf{saut}(\omega_2)$ be simple. Then Lie subalgebra $\mathfrak{g} = \mathbf{aut}(\omega_1) \oplus \mathbf{aut}(\omega_2)$ is maximal in $\mathbf{aut}(\omega_1 \otimes \omega_2)$ if $p(\omega_1) + p(\omega_2) = \bar{0}$ or if $p(\omega_i) = \bar{0}$, $p(\omega_j) = \bar{1}$ and $\dim(V_i)_{\bar{0}} \neq \dim(V_i)_{\bar{1}}$ for $i, j = 1, 2$. If $p(\omega_i) = \bar{0}$, $p(\omega_j) = \bar{1}$ and $\dim(V_i)_{\bar{0}} = \dim(V_i)_{\bar{1}}$, then \mathfrak{g} is maximal in $\mathbf{saut}(\omega_1 \otimes \omega_2)$.*

For exact answer in the standard format see table from Th. 0.2.4.

Proof. Let \mathfrak{f} be a Lie superalgebra such that $\mathfrak{g} \subset \mathfrak{f} \subset \mathbf{aut}(\omega_1 \otimes \omega_2)$ and $h \in \mathfrak{f}$ but $h \notin \mathfrak{g}$. It follows from (4.3) that

$$h = \sum A_i^a \otimes B_i^s + \sum A_j^s \otimes B_j^a$$

for some $A_i^a \in \mathbf{aut}(\omega_1)$, $B_i^s \in \mathbf{symm}(\omega_2)$, $A_j^s \in \mathbf{symm}(\omega_1)$, $B_j^a \in \mathbf{aut}(\omega_2)$. Let us bracket h with elements of \mathfrak{g} :

$$\begin{aligned} [C \otimes 1, h] &= \sum [C, A_i^a] \otimes B_i^s + \sum [C, A_j^s] \otimes B_j^a \\ [1 \otimes D, h] &= \sum (-1)^{p(D)p(A_i^a)} A_i^a \otimes [D, B_i^s] + \sum (-1)^{p(D)p(A_j^s)} A_j^s \otimes [D, B_j^a]. \end{aligned}$$

Since $\mathbf{saut}(\omega_1)$ and $\mathbf{saut}(\omega_2)$ are simple and (as follows from [K]) the sole $\mathbf{aut}(\omega)$ -invariant superspaces in $\mathbf{symm}(\omega)$ are the space of constants and $\mathbf{ssym}(\omega)$, it follows that

$$\mathfrak{f} \supset (\mathbf{saut}(\omega_1) \otimes \mathbf{ssym}(\omega_2)) \oplus (\mathbf{ssym}(\omega_1) \otimes \mathbf{saut}(\omega_2)) = \mathfrak{a} \oplus \mathfrak{b}.$$

Now notice that $\mathfrak{sl}(V)$ is not closed with respect to $\{\cdot, \cdot\}$.

Let $p(\omega_1) = p(\omega_2) = \bar{0}$. Then $\mathbf{aut}(\omega_i) = \mathbf{saut}(\omega_i)$ for $i = 1, 2$. Let $A \otimes B, C \otimes D \in \mathfrak{a}$. We have:

$$[A, C] \otimes \{B, D\} \in \mathbf{aut}(\omega_1) \otimes \mathbf{symm}(\omega_2), \quad \{A, C\} \otimes [B, D] \in \mathbf{symm}(\omega_1) \otimes \mathbf{aut}(\omega_2).$$

Since $\mathfrak{sl}(V)$ is not closed with respect to $\{\cdot, \cdot\}$ the above formulas imply that

$$\mathfrak{f} \supset \mathbf{aut}(\omega_1) \otimes \mathbf{symm}(\omega_2) \oplus \mathbf{symm}(\omega_1) \otimes \mathbf{aut}(\omega_2) = \mathbf{aut}(\omega_1 \otimes \omega_2).$$

Let $p(\omega_1) = p(\omega_2) = \bar{1}$ and, therefore, $\mathbf{ssym}(\omega_i) = \mathbf{symm}(\omega_i)$. Let $A \otimes B \in \mathfrak{a}, C \otimes D \in \mathfrak{b}$. Then

$$[A, C] \otimes \{B, D\} \in \mathbf{symm}(\omega_1) \otimes \mathbf{aut}(\omega_2), \quad \{A, C\} \otimes [B, D] \in \mathbf{aut}(\omega_1) \otimes \mathbf{symm}(\omega_2)$$

which means that $\mathfrak{f} = \mathbf{aut}(\omega_1 \otimes \omega_2)$.

Finally, let $p(\omega_1) = \bar{0}, p(\omega_2) = \bar{1}$. Then $\mathbf{saut}(\omega_1) \otimes \mathbf{ssym}(\omega_2) = \mathbf{aut}(\omega_1) \otimes \mathbf{symm}(\omega_2)$. Taking the bracket of $A \otimes B \in \mathfrak{a}$ with $C \otimes D \in \mathfrak{b}$, we see that

$$\mathfrak{f} \supset \mathbf{aut}(\omega_1) \otimes \mathbf{symm}(\omega_2) \oplus \mathbf{ssym}(\omega_1) \otimes \mathbf{aut}(\omega_2)$$

and taking the bracket of $A \otimes B, C \otimes D \in \mathfrak{b}$ we get

$$\mathfrak{f} \supset \mathfrak{aut}(\omega_1) \otimes \text{symm}(\omega_2) + \text{symm}(\omega_1) \otimes \mathfrak{saut}(\omega_2).$$

These two inclusions together mean that $\mathfrak{f} \supset \mathfrak{saut}(\omega_1 \otimes \omega_2)$. This completes the proof when $\dim(V_1)_{\bar{0}} = \dim(V_1)_{\bar{1}}$, while for $\dim(V_1)_{\bar{0}} \neq \dim(V_1)_{\bar{1}}$ it suffices to observe that \mathfrak{g} is not contained in $\mathfrak{sl}(V)$. \square

APPENDIX 1: PROOF OF LEMMA 2.5

Let $\dim \mathfrak{r} = 0$, i.e., \mathfrak{g} is semisimple. As \mathfrak{g} is not almost simple, then due to sec. A2.6 the alternative arises: either \mathfrak{g} contains an ideal \mathfrak{k} of the form

$$\mathfrak{k} = \mathfrak{h} \otimes \Lambda(n) \quad \text{with simple } \mathfrak{h} \text{ and } n > 0 \quad (\text{A1.1})$$

or $\mathfrak{g} = \bigoplus_{j \leq k} \mathfrak{s}_j$, where each \mathfrak{s}_j is almost simple and $k > 1$.

A1.1. Lemma . *If $\mathfrak{g} = \bigoplus_{j \leq k} \mathfrak{s}_j$ and $k \geq 2$, then any irreducible faithful representation of \mathfrak{g} is of type B with respect to any its ideal \mathfrak{s}_j .*

Proof. Since the stabilizer of any irreducible representation of \mathfrak{s}_j is the whole \mathfrak{g} , the type of any irreducible representation of \mathfrak{g} with respect to \mathfrak{s}_j can be either A or B, see sec. 2.0. Due to faithfulness case A is ruled out. \square

For \mathfrak{k} take $\mathfrak{asi} \otimes \Lambda(n)$.

A1.2. Lemma . *Let \mathfrak{h} be a simple Lie superalgebra and $\mathfrak{k} = \mathfrak{h} \otimes \Lambda(n)$. Then \mathfrak{k} has no faithful irreducible finite dimensional representations.*

For proof see A1.4.

A1.2.1. Corollary . *If \mathfrak{g} contains an ideal \mathfrak{k} of the form (A1.1), then \mathfrak{g} can not have any faithful irreducible finite dimensional representation of type A with respect to the ideal \mathfrak{k} .*

A1.2.2. Corollary . *Lemma A2.1 and Corollary A2.2.1 prove Lemma 2.5 for semisimple Lie superalgebras.*

A1.3. Lemma . *If \mathfrak{h} is a simple Lie superalgebra, then*

$$[\mathfrak{h}_{\bar{1}}, \mathfrak{h}_{\bar{1}}] = \mathfrak{h}_{\bar{0}} \quad \text{and} \quad [\mathfrak{h}_{\bar{0}}, \mathfrak{h}_{\bar{1}}] = \mathfrak{h}_{\bar{1}}.$$

Proof is left to the reader. (*Hint:* show that $[\mathfrak{h}_{\bar{1}}, \mathfrak{h}_{\bar{1}}] \oplus \mathfrak{h}_{\bar{1}}$ and $\mathfrak{h}_{\bar{0}} \oplus [\mathfrak{h}_{\bar{0}}, \mathfrak{h}_{\bar{1}}]$ are ideals in \mathfrak{h} .) \square

A1.4. Proof of Lemma A1.2. For $n > 0$ the Lie superalgebra \mathfrak{g} contains supercommutative ideal

$$\mathfrak{n}_1 = \mathfrak{h} \otimes \xi_1 \dots \xi_n.$$

Moreover, if $n > 2$, then \mathfrak{g} additionally contains a supercommutative ideal

$$\mathfrak{n}_2 = \mathfrak{h} \otimes \Lambda^{n-1}(\xi) \oplus \mathfrak{n}_1.$$

Thanks to Theorem 2.0, any irreducible representation ρ of \mathfrak{k} is obtained as follows: take an irreducible representation (character χ) of an ideal \mathfrak{n}_i , the subsuperalgebra $\mathfrak{st}_{\mathfrak{k}}(\chi)$, an irreducible representation $\tilde{\rho}$ of $\mathfrak{st}_{\mathfrak{k}}(\chi)$ whose restriction onto \mathfrak{n}_i is a multiple of χ . Then $\rho = \text{ind}_{\mathfrak{st}_{\mathfrak{k}}(\chi)}^{\mathfrak{k}}(\tilde{\rho})$ for some ρ .

Let us show that if $\dim \rho < \infty$ then $\chi|_{\mathfrak{h} \otimes \xi_1 \dots \xi_n} = 0$. Since $\mathfrak{h} \otimes \xi_1 \dots \xi_n$ is an ideal in \mathfrak{g} , this would contradict the faithfulness of ρ . Indeed, $\dim \rho < \infty$ if and only if $\mathfrak{st}_{\mathfrak{k}}(\chi) \supset \mathfrak{k}_{\bar{0}}$.

1) $n = 2k + 1$. For any $f \in \mathfrak{h}$ and $g = h \otimes 1 \in \mathfrak{h}_{\bar{0}} \otimes 1$ we have $0 = \chi([g, f \otimes \xi_1 \dots \xi_n]) = \chi([h, f] \otimes \xi_1 \dots \xi_n)$ implying that

$$\chi|_{[\mathfrak{h}_{\bar{0}}, \mathfrak{h}_{\bar{1}}] \otimes \xi_1 \dots \xi_n} \stackrel{(\text{Lemma A1.3})}{=} \chi|_{(\mathfrak{h}_{\bar{1}} \otimes \xi_1 \dots \xi_n)} = 0.$$

Since χ is a character, then $\chi|_{\mathfrak{h}_{\bar{0}} \otimes \xi_1 \dots \xi_n} = 0$, i.e., $\chi|_{\mathfrak{h} \otimes \xi_1 \dots \xi_n} = 0$.

2) $n = 2k > 2$. For any

$$\begin{aligned} g &= h \otimes l_1(\xi) \in \mathfrak{h}_{\bar{1}} \otimes \Lambda^1(\xi) \subset \mathfrak{g}_{\bar{0}} \\ g_1 &= h_1 \otimes l_{n-1}(\xi) \in \mathfrak{h}_{\bar{1}} \otimes \Lambda^{n-1}(\xi) \subset (\mathfrak{n}_2)_{\bar{0}} \end{aligned}$$

we have

$$0 = \chi([g, g_1]) = \pm \chi([h, h_1]) \otimes l_1(\xi) l_{n-1}(\xi),$$

i.e.,

$$\chi|_{[\mathfrak{h}_{\bar{1}}, \mathfrak{h}_{\bar{1}}] \otimes \xi_1 \dots \xi_n} = \chi|_{\mathfrak{h}_{\bar{0}} \otimes \xi_1 \dots \xi_n} = 0.$$

But χ is a character; therefore,

$$\chi|_{\mathfrak{h}_1 \otimes \xi_1 \cdots \xi_n} = 0$$

and, finally,

$$\chi|_{\mathfrak{h} \otimes \xi_1 \cdots \xi_n} = 0. \quad \square$$

3) $n = 2$. Then \mathfrak{k} contains a commutative ideal $\mathfrak{n} = \mathfrak{h} \otimes \xi_1 \xi_2$; let χ be a character of \mathfrak{n} .

If χ generates a finite dimensional representation of \mathfrak{g} then $\mathfrak{st}_{\mathfrak{g}}(\chi) \supset \mathfrak{g}_{\bar{0}}$, i.e.,

$$\chi([f, g \otimes \xi_1 \xi_2]) = 0 \text{ for any } f \in \mathfrak{h}_{\bar{0}}, g \in \mathfrak{h}.$$

If $g \in \mathfrak{h}_{\bar{1}}$, then formula (A1.1) holds automatically by the parity considerations. Let $g \in \mathfrak{h}_{\bar{0}}$. Then, nevertheless,

$$\chi([f, g \otimes \xi_1 \xi_2]) = \chi([f, g] \otimes \xi_1 \xi_2) = 0$$

and, therefore, $\chi|_{[\mathfrak{h}_{\bar{0}}, \mathfrak{h}_{\bar{0}}] \otimes \xi_1 \xi_2} = 0$.

Set

$$\mathfrak{m} = \mathfrak{h} \otimes (\Lambda^1(\xi) \oplus \Lambda^2(\xi)).$$

This is an ideal of \mathfrak{g} and $\mathfrak{st}_{\mathfrak{g}}(\chi) = \mathfrak{m}$. Therefore, the irreducible representation ρ of \mathfrak{m} given by χ is such that $\rho|_{\mathfrak{n}}$ is a multiple of χ . Let

$$\mathfrak{n}_1 = \mathfrak{h}_{\bar{0}} \otimes \Lambda^1(\xi) \oplus ([\mathfrak{h}_{\bar{0}}, \mathfrak{h}_{\bar{0}}] \oplus \mathfrak{h}_{\bar{1}}) \otimes \Lambda^2(\xi).$$

This is an ideal in \mathfrak{m} and

$$[\mathfrak{n}_1, \mathfrak{m}] \subset ([\mathfrak{h}_{\bar{0}}, \mathfrak{h}_{\bar{0}}] \oplus \mathfrak{h}_{\bar{1}}) \otimes \Lambda^2(\xi) \subset \ker \chi.$$

Thus, $\mathfrak{n}_1 \subset \ker \rho$ since the subspace $V^{\rho(\mathfrak{n}_1)} = \{v \in V : \rho(\mathfrak{n}_1)v = 0\}$ is $\rho(\mathfrak{m}_1)$ -invariant and non-zero. But $\mathfrak{m}/\mathfrak{n}_1$ is a solvable Lie algebra and by the Lie theorem any its irreducible finite dimensional representation is of the same form as χ . Let it be χ , for definiteness sake.

This means that $\chi|_{[\mathfrak{h}_{\bar{1}}, \mathfrak{h}_{\bar{1}}] \otimes \Lambda^2(\xi)} = 0$, i.e., $\chi|_{\mathfrak{h} \otimes \Lambda^2(\xi)} = 0$ and since $\mathfrak{h} \otimes \xi_1 \xi_2$ is an ideal in \mathfrak{k} , then the representation of \mathfrak{k} given by χ cannot be faithful. Lemma A1.2 is proved. \square

A1.5. Central extensions of semisimple Lie superalgebras. Let \mathfrak{g} be a Lie superalgebra, \mathfrak{r} its radical of dimension 1. Then \mathfrak{r} is the center of \mathfrak{g} and $\tilde{\mathfrak{g}} = \mathfrak{g}/\mathfrak{r}$ is semisimple.

First, consider the case when $\tilde{\mathfrak{g}} = \bigoplus_{i \leq k} \mathfrak{s}_i$, where the \mathfrak{s}_i are almost simple and $k > 1$. Let $\pi : \mathfrak{g} \rightarrow \tilde{\mathfrak{g}}$ be the natural projection. The Lie superalgebra $\pi^{-1}(\mathfrak{s}_1) = \mathfrak{k}$ is an ideal in \mathfrak{g} and $\dim \mathfrak{k} > 1$.

A1.5.1. Lemma . *If $k > 1$, then ρ can not be irreducible of type A with respect to \mathfrak{k} .*

Proof. Set $\mathfrak{g}_+ = \bigoplus_{i \geq 1} (\mathfrak{s}_i)_{\bar{0}}$ and \mathfrak{g}_+ is a Lie algebra. Then $[\mathfrak{s}_1, \mathfrak{g}_+] = 0$. Since $\rho(\mathfrak{r})$ acts by scalar operators and ρ is a finite dimensional representation, it follows that $[\pi^{-1}(\mathfrak{g}_+), \mathfrak{k}] = 0$, i.e., $\rho(\pi^{-1}(\mathfrak{g}_+))$ consists of intertwining operators of $\rho|_{\mathfrak{n}}$, i.e., $\rho(\pi^{-1}(\mathfrak{g}_+)) = \mathbb{C} \cdot 1$ and, since ρ is faithful, then $\pi^{-1}(\mathfrak{g}_+) = \mathfrak{r}$ implying $\mathfrak{g}_+ = 0$. \square

A1.5.2. Now, let $\tilde{\mathfrak{g}}$ contain an ideal \mathfrak{k} of the form (A1.1). The central extension is defined by a cocycle $c : \tilde{\mathfrak{g}} \times \tilde{\mathfrak{g}} \rightarrow \mathbb{C}$. The cocycle condition is

$$c(f, [g, h]) = c([f, g], h) + (-1)^{p(f)p(g)} c(g, [f, h]) \text{ for any } f, g, h \in \tilde{\mathfrak{g}}.$$

As earlier, we assume that \mathfrak{g} has a faithful finitedimensional representation; so the restriction of c to $\tilde{\mathfrak{g}}_{\bar{0}} \times \tilde{\mathfrak{g}}_{\bar{0}}$ is trivial. Besides, $c|_{\tilde{\mathfrak{g}}_{\bar{0}} \times \tilde{\mathfrak{g}}_{\bar{1}}} = 0$ by parity considerations. Therefore,

$$c|_{\tilde{\mathfrak{g}}_{\bar{0}} \times \tilde{\mathfrak{g}}} = 0. \quad (\text{A1.2})$$

Lemma . *Set $\mathfrak{L}^m = \mathfrak{h} \otimes \Lambda^m(\xi_1, \dots, \xi_n)$. We have $c|_{\mathfrak{L}^n \times \mathfrak{L}^n} = 0$.*

Proof. If $n = 2k + 1$ the condition (A1.2) means that

$$c|_{\mathfrak{h}_1 \otimes \Lambda^n(\xi) \times \mathfrak{h}_1 \otimes \Lambda^n(\xi)} = 0.$$

Let $g_{\bar{1}}, h_{\bar{1}} \in \mathfrak{h}_{\bar{1}}$ and $h_{\bar{0}} \in \mathfrak{h}_{\bar{0}}$. Then

$$\begin{aligned} c(f_{\bar{0}} \otimes \xi_1 \cdots \xi_n, [g_{\bar{1}}, h_{\bar{1}}] \otimes \xi_1 \cdots \xi_n) &= -c(f_{\bar{0}} \otimes \xi_1 \cdots \xi_n, [g_{\bar{1}} \otimes \xi_1 \cdots \xi_n, h_{\bar{1}}]) = \\ &= -c([f_{\bar{0}} \otimes \xi_1 \cdots \xi_n, g_{\bar{1}} \otimes \xi_1 \cdots \xi_n], h_{\bar{1}}) - c(g_{\bar{1}} \otimes \xi_1 \cdots \xi_n, [f_{\bar{0}} \otimes \xi_1 \cdots \xi_n, h_{\bar{1}}]) = 0. \end{aligned}$$

Since $\mathfrak{h}_{\bar{0}} = [\mathfrak{h}_{\bar{1}}, \mathfrak{h}_{\bar{1}}]$ (by Lemma A1.3), it follows that $c|_{\mathfrak{L}^n \times \mathfrak{L}^n} = 0$.

If $n = 2k$ the condition (A1.2) means that

$$c|_{\mathfrak{h}_{\bar{0}} \otimes \Lambda^n(\xi) \times \mathfrak{h}_{\bar{0}} \otimes \Lambda^n(\xi)} = 0.$$

Let $f_{\bar{1}}, h_{\bar{1}} \in \mathfrak{h}_{\bar{1}}$ and $g_{\bar{0}} \in \mathfrak{h}_{\bar{0}}$. Then

$$\begin{aligned} c(f_{\bar{1}} \otimes \xi_1 \cdots \xi_n, [g_{\bar{0}}, h_{\bar{1}}] \otimes \xi_1 \cdots \xi_n) &= -c(f_{\bar{1}} \otimes \xi_1 \cdots \xi_n, [g_{\bar{0}} \otimes \xi_1 \cdots \xi_n, h_{\bar{1}}]) = \\ &= -c([f_{\bar{1}} \otimes \xi_1 \cdots \xi_n, g_{\bar{0}}] \otimes \xi_1 \cdots \xi_n, h_{\bar{1}}) - c(g_{\bar{0}} \otimes \xi_1 \cdots \xi_n, [f_{\bar{1}} \otimes \xi_1 \cdots \xi_n, h_{\bar{1}}]) = 0. \end{aligned}$$

Since $\mathfrak{h}_{\bar{1}} = [\mathfrak{h}_{\bar{0}}, \mathfrak{h}_{\bar{1}}]$ (by Lemma A1.3), it follows that $c|_{\mathfrak{L}^n \times \mathfrak{L}^n} = 0$. \square

A1.5.3. Lemma . $c|_{\mathfrak{L}^k \times \mathfrak{L}^m} = 0$ for any $m = 0, 1, \dots, n$ and $k = n - m, n - m + 1, \dots, n$.

Proof. We will perform the inverse double induction on k and m . For $k = m = n$ Lemma A1.5.2 will do.

Let the statement be true for all $k > k_0$ and $k_0 + m \geq n$. Let us show that it is true for $k = k_0$ as well. Observe that due to sec. A2.6, \mathfrak{g} contains n elements η_i such that $\text{ad}\eta_i|_{\mathfrak{h}} = \partial_{\xi_i} + D_i$, where $D_i \in \mathbf{vect}(0|n)$ and $\deg D_i > 0$ in the natural grading of \mathbf{vect} . Let $\varphi \in \Lambda^{k_0+1}(\xi_1, \dots, \xi_n)$, $\psi \in \Lambda^m(\xi_1, \dots, \xi_n)$; $g, h \in \mathfrak{h}$ and $p(g \otimes \psi) = \bar{0}$; $p(h \otimes \psi) = \bar{1}$.

Then we have

$$\begin{aligned} c([\eta_i, g \otimes \varphi], h \otimes \psi) &= (-1)^{p(g)} c(g \otimes \eta_i(\varphi), h \otimes \psi) \\ &= (-1)^{p(g)} \left(c(g \otimes \frac{\partial \varphi}{\partial \xi_i}, h \otimes \psi) + c(g \otimes D_i \varphi, h \otimes \psi) \right). \end{aligned}$$

As to $D_i \varphi \in \bigoplus_{s \geq k_0+1} \Lambda^s(\xi)$, the last summand is equal to zero by the inductive hypothesis.

On the other hand,

$$c([\eta_i, g \otimes \varphi], h \otimes \psi) = c(\eta_i, [g \otimes \varphi, h \otimes \psi]) + c([\eta_i, h \otimes \psi], g \otimes \varphi).$$

As $\deg \varphi + \deg \psi = k_0 + 1 + m > n$, the bracket in the first summand above is equal to 0. The second summand is equal to 0 by (A1.2). So $c(g \otimes \frac{\partial \varphi}{\partial \xi_i}, h \otimes \psi) = 0$, for arbitrary $g \otimes \frac{\partial \varphi}{\partial \xi_i} \in \mathfrak{L}_1^{k_0}$ and $h \otimes \psi \in \mathfrak{L}_1^m$. \square

A1.5.4. Lemma . \mathfrak{k} has no faithful irreducible finite dimensional representations.

Proof. Word-for-word proof of Lemmas A1.2.1 with the help of Lemmas A1.5.2 and A1.5.3. \square

Corollary . ρ can not be of type A with respect to the ideal \mathfrak{k} .

APPENDIX 2: BACKGROUND

A2.1. Linear algebra in superspaces. A *superspace* is a $\mathbb{Z}/2$ -graded space; for a superspace $V = V_{\bar{0}} \oplus V_{\bar{1}}$ denote by $\Pi(V)$ another copy of the same superspace: with the shifted parity, i.e., $\Pi(V_{\bar{i}}) = V_{\bar{i}+\bar{1}}$.

A superspace structure in V induces the superspace structure in the space $\text{End}(V)$. A *basis of a superspace* is always a basis consisting of *homogeneous* vectors; let $Par = (p_1, \dots, p_{\dim V})$ be an ordered collection of their parities, called the *format* of V .

A *supermatrix* of format (size) Par is a $\dim V \times \dim V$ matrix whose i th row and column are said to be of parity p_i . The matrix unit E_{ij} is supposed to be of parity $p_i + p_j$ and the bracket of supermatrices (of the same format) is defined via Sign Rule:

if something of parity p moves past something of parity q the sign $(-1)^{pq}$ accrues; the formulas defined on homogeneous elements are extended to arbitrary ones via linearity.

For example: $[X, Y] = XY - (-1)^{p(X)p(Y)} YX$; the sign \wedge of the exterior or wedge product is also understood in this text in the supersence, etc.

Usually, Par is of the form $(\bar{0}, \dots, \bar{0}, \bar{1}, \dots, \bar{1})$. Such a format is called *standard*.

The *general linear* Lie superalgebra of all supermatrices of size Par is denoted by $\mathfrak{gl}(Par)$, usually $\mathfrak{gl}(\bar{0}, \dots, \bar{0}, \bar{1}, \dots, \bar{1})$ is abbreviated to $\mathfrak{gl}(\dim V_{\bar{0}} | \dim V_{\bar{1}})$.

The supercommutative superalgebra with unit generated by odd indeterminates $\theta_1, \dots, \theta_n$ is called Grassmann superalgebra and is denoted by $\Lambda(n)$ or $\Lambda(\theta)$. The Lie superalgebra of superderivations of $\mathbb{C}[x, \theta]$ is denoted by $\mathbf{vect}(m|n)$. Clearly, every element from $\mathbf{vect}(m|n)$ is of the form

$$D = \sum_{i=1}^m f_i(x, \theta) \frac{\partial}{\partial x_i} + \sum_{j=1}^n g_i(x, \theta) \frac{\partial}{\partial \theta_j}.$$

Define the *divergence* of a vector field setting

$$\text{div} \left(\sum_{i=1}^m f_i(x, \theta) \frac{\partial}{\partial x_i} + \sum_{j=1}^n g_i(x, \theta) \frac{\partial}{\partial \theta_j} \right) = \sum_{i=1}^m \frac{\partial f_i}{\partial x_i} + \sum_{j=1}^n (-1)^{p(g_i)} \frac{\partial g_i}{\partial \theta_j}.$$

The Lie superalgebra of divergence-free vector fields is denoted by $\mathbf{svect}(m|n)$.

A2.2. General linear superalgebras and irreducibilities: two types. The Lie superalgebra $\mathfrak{gl}(m|n)$ is called the *general Lie superalgebra*. Its supermatrices (in the standard format) can be expressed as the sum of the even and odd parts:

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} + \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix}.$$

The *supertrace* is the map $\mathfrak{gl}(Par) \rightarrow \mathbb{C}$, $(A_{ij}) \mapsto \sum (-1)^{p_i} A_{ii}$. The space of supertraceless matrices constitutes the *special linear* Lie subsuperalgebra $\mathfrak{sl}(Par)$.

There are, however, two super versions of $\mathfrak{gl}(n)$, not one. The other version is called the *queer* Lie superalgebra and is defined as the one that preserves the complex structure given by an *odd* operator J , i.e., is the centralizer $C(J)$ of J :

$$\mathfrak{q}(n) = C(J) = \{X \in \mathfrak{gl}(n|n) : [X, J] = 0\}, \text{ where } J^2 = -\text{id}.$$

It is clear that by a change of basis we can reduce J to the form $J_{2n} = \begin{pmatrix} 0 & 1_n \\ -1 & 0 \end{pmatrix}$. In the standard format we have

$$\mathfrak{q}(n) = \left\{ \begin{pmatrix} A & B \\ B & A \end{pmatrix} \right\}.$$

On $\mathfrak{q}(n)$, the *queer trace* is defined: $\text{qtr} : \begin{pmatrix} A & B \\ B & A \end{pmatrix} \mapsto \text{tr} B$. Denote by $\mathfrak{sq}(n)$ the Lie superalgebra of *queertraceless* matrices.

Observe that the identity representations of \mathfrak{q} and \mathfrak{sq} in V , though irreducible in supersetting, are not irreducible in the nongraded sense: take homogeneous linearly independent vectors v_1, \dots, v_n from V ; then $\text{Span}(v_1 + J(v_1), \dots, v_n + J(v_n))$ is an invariant subspace of V which is not a subsuperspace.

A representation is called *irreducible of G-type* if it has no invariant subspace; it is called *irreducible of Q-type* if it has no invariant subsuperspace, but has an invariant subspace.

A2.3. Lie superalgebras that preserve bilinear forms: two types. To the linear map F of superspaces there corresponds the dual map F^* between the dual superspaces; if A is the supermatrix corresponding to F in a format Par , then to F^* the *supertransposed* matrix A^{st} corresponds:

$$(A^{st})_{ij} = (-1)^{(p_i+p_j)(p_i+p(A))} A_{ji}.$$

The supermatrices $X \in \mathfrak{gl}(Par)$ such that

$$X^{st}B + (-1)^{p(X)p(B)}BX = 0 \quad \text{for a homogeneous matrix } B \in \mathfrak{gl}(Par)$$

constitute a Lie superalgebra $\mathfrak{aut}(B)$ that preserves a homogeneous bilinear form on V with matrix B .

Recall that the *supersymmetry* of the homogeneous form ω means that its matrix F satisfies the condition $F^u = F$, where $F^u = \begin{pmatrix} R_n^t & -T^t \\ -S^t & -U_m^t \end{pmatrix}$ for the matrix $F = \begin{pmatrix} R_n & S \\ T & U_m \end{pmatrix}$, where the subscripts n and m indicate the size of the square matrices.

A nondegenerate supersymmetric even bilinear form can be reduced to a canonical form whose matrix in the standard format is

$$B_{ev}(m|2n) = \begin{pmatrix} 1_m & 0 \\ 0 & J_{2n} \end{pmatrix}, \quad \text{where } J_{2n} = \begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix},$$

or

$$B'_{ev}(m|2n) = \begin{pmatrix} \text{antidiag}(1, \dots, 1) & 0 \\ 0 & J_{2n} \end{pmatrix}.$$

The usual notation for $\mathfrak{aut}(B_{ev}(Par))$ in the standard format is $\mathfrak{osp}(m|2n) = \mathfrak{osp}^{sy}(m|2n)$. (Observe that the passage from V to $\Pi(V)$ sends the supersymmetric forms to superskew-symmetric ones, preserved by $\mathfrak{sp}'\mathfrak{o}(2n|m) \cong \mathfrak{osp}^{sk}(m|2n)$.)

In the standard format the matrix realizations of these algebras are:

$$\mathfrak{osp}(m|2n) = \left\{ \begin{pmatrix} E & Y & -X^t \\ X & A & B \\ Y^t & C & -A^t \end{pmatrix} \right\}; \quad \mathfrak{osp}^{sk}(2n|m) = \left\{ \begin{pmatrix} A & B & X \\ C & -A^t & Y^t \\ Y^t & -X^t & E \end{pmatrix} \right\},$$

where $\begin{pmatrix} A & B \\ C & -A^t \end{pmatrix} \in \mathfrak{sp}(2n)$ and $E \in \mathfrak{o}(m)$.

A nondegenerate supersymmetric odd bilinear form $B_{odd}(n|n)$ can be reduced to a canonical form whose matrix in the standard format is J_{2n} . A canonical form of the superskew odd nondegenerate form in the

standard format is $\Pi_{2n} = \begin{pmatrix} 0 & 1_n \\ 1_n & 0 \end{pmatrix}$. The usual notation for $\mathfrak{aut}(B_{\text{odd}}(\text{Par}))$ is $\mathfrak{pe}^{sy}(n)$. The passage from V to $\Pi(V)$ sends the supersymmetric forms to superskew-symmetric ones and establishes an isomorphism $\mathfrak{pe}^{sy}(2n|m) \cong \mathfrak{pe}^{sk}(n)$. This Lie superalgebra is called *periplectic*. The matrix realizations in the standard format of these superalgebras is:

$$\begin{aligned} \mathfrak{pe}^{sy}(n) &= \left\{ \begin{pmatrix} A & B \\ C & -A^t \end{pmatrix}, \text{ where } B = B^t, C = -C^t; \right\}; \\ \mathfrak{pe}^{sk}(n) &= \left\{ \begin{pmatrix} A & B \\ C & -A^t \end{pmatrix}, \text{ where } B = -B^t, C = C^t \right\}. \end{aligned}$$

The *special periplectic* superalgebra is $\mathfrak{spe}(n) = \{X \in \mathfrak{pe}(n) : \text{str} X = 0\}$.

A2.4. A. Sergeev's central extension. A. Sergeev proved that there is just one nontrivial central extension of $\mathfrak{spe}(n)$. It exists only for $n = 4$ and is denoted by \mathfrak{as} . Let us represent an arbitrary element $A \in \mathfrak{as}$ as a pair $A = x + d \cdot z$, where $x \in \mathfrak{spe}(4)$, $d \in \mathbb{C}$ and z is the central element. The bracket in \mathfrak{as} in the matrix form is

$$\left[\begin{pmatrix} a & b \\ c & -a^t \end{pmatrix} + d \cdot z, \begin{pmatrix} a' & b' \\ c' & -a'^t \end{pmatrix} + d' \cdot z \right] = \left[\begin{pmatrix} a & b \\ c & -a^t \end{pmatrix}, \begin{pmatrix} a' & b' \\ c' & -a'^t \end{pmatrix} \right] + \text{tr } cc' \cdot z.$$

A2.5. Projectivization. If \mathfrak{s} is a Lie algebra of scalar matrices, and $\mathfrak{g} \subset \mathfrak{gl}(n|n)$ is a Lie subsuperalgebra containing \mathfrak{s} , then the *projective* Lie superalgebra of type \mathfrak{g} is $\mathfrak{pg} = \mathfrak{g}/\mathfrak{s}$. Lie superalgebras $\mathfrak{g}_1 \odot \mathfrak{g}_2$ described in Introduction are also projective.

Projectivization sometimes leads to new Lie superalgebras, for example: $\mathfrak{pgl}(n|n)$, $\mathfrak{psl}(n|n)$, $\mathfrak{pq}(n)$, $\mathfrak{psq}(n)$; whereas $\mathfrak{pgl}(p|q) \cong \mathfrak{sl}(p|q)$ if $p \neq q$.

A2.6. Simplicity. The Lie superalgebras $\mathfrak{sl}(m|n)$ for $m > n \geq 1$, $\mathfrak{psl}(n|n)$ for $n > 1$, $\mathfrak{psq}(n)$ for $n > 2$, $\mathfrak{osp}(m|2n)$ for $mn \neq 0$ and $\mathfrak{spe}(n)$ for $n > 2$ are simple.

We say that \mathfrak{h} is *almost simple* if it can be included (nonstrictly) between a simple Lie superalgebra \mathfrak{s} and the Lie superalgebra $\mathfrak{der} \mathfrak{s}$ of derivations of the latter: $\mathfrak{s} \subset \mathfrak{h} \subset \mathfrak{der} \mathfrak{s}$.

By definition \mathfrak{g} is *semisimple* if its radical is zero. V. Kac [K] described semisimple Lie superalgebras. Let $\mathfrak{s}_1, \dots, \mathfrak{s}_k$ be simple Lie superalgebras, let n_1, \dots, n_k be *pairs* of nonnegative integers $n_j = (n_j^0, n_j^1)$, $\mathcal{F}(n_j)$ be the supercommutative superalgebra of polynomials of n_j^0 even and n_j^1 odd variables, and $\mathfrak{s} = \oplus \mathfrak{s}_j \otimes \mathcal{F}(n_j)$. Then $\mathfrak{ders} = \oplus ((\mathfrak{ders}_j) \otimes \mathcal{F}(n_j) \oplus 1 \otimes \mathfrak{vect}(n_j))$. Let \mathfrak{g} be a subalgebra of \mathfrak{ders} containing \mathfrak{s} .

1) If the projection of \mathfrak{g} on $1 \otimes \mathfrak{vect}(n_j)_{-1}$ coincides with $\mathfrak{vect}(n_j)_{-1}$ for each $j = 1, \dots, k$, then \mathfrak{g} is semisimple.

2) All semisimple Lie superalgebras arise in the manner indicated. For finite dimensional \mathfrak{g} all the ingredients must be also of finite dimension, e.g., $n_j^0 = 0$ for all j .

A2.7. Lemma . 1) Let \mathfrak{g} be an irreducible (in V) finite dimensional matrix Lie superalgebra. Then the space \mathcal{B} of \mathfrak{g} -invariant bilinear forms on V is at most 1-dimensional.

2) If $\dim \mathcal{B} = 1$ or ε , then \mathfrak{g} is G -irreducible.

Proof. Since the kernel of any \mathfrak{g} -invariant form is a \mathfrak{g} -invariant subspace, every \mathfrak{g} -invariant form is either nondegenerate or identically zero.

Let $\mathcal{B} \neq 0$; let $B \in \mathcal{B}$ be a nondegenerate form. We denote its Gram matrix also by B . The invariance of B means that

$$X^{st} B + (-1)^{p(B)p(X)} B X = 0 \quad \text{for any } X \in \mathfrak{g}, \text{ or } X^{st} = -(-1)^{p(B)p(X)} B X B^{-1}. \quad (\text{A2.1})$$

For another form $C \in \mathcal{B}$ we have from (A2.1):

$$(-1)^{p(B)p(X)} B X B^{-1} C + (-1)^{p(C)p(X)} C X = 0$$

implying that X preserves the operator with matrix $B^{-1}C$.

If \mathfrak{g} is G -irreducible, this immediately implies that $B^{-1}C = c \cdot \text{id}$, i.e., $B = c \cdot C$ for a constant c .

If \mathfrak{g} is Q -irreducible, then $B^{-1}C \in \text{Span}(\text{id}, J)$; in particular, $\dim \mathcal{B}_0 \leq 1$ and $\dim \mathcal{B}_1 \leq 1$. Denote by sign the operator with matrix $\text{diag}(1_n, -1_m)$ in the standard format; i.e., id on the even subspace and $-\text{id}$ on the odd one. Supertransposing formula (A2.1) and taking into account that $(X^{st})^2 = \text{sign} \cdot X \cdot \text{sign}$, we see that the form $B^{st} \text{sign}$ is \mathfrak{g} -invariant. Having multiplied (A2.1) by J^{st} from the left and by J from the right and taking into account that $[X, J] = 0$ for any $X \in \mathfrak{g}$, we see that the form JB is also \mathfrak{g} -invariant. All the three forms: B , $B^{st} \text{sign}$ and JB are of the same parity and, therefore, are proportional.

If $p(B) = \bar{0}$, then

$$B = \begin{pmatrix} B_0 & 0 \\ 0 & B_2 \end{pmatrix}, \quad B^{st} \text{ sign} = \begin{pmatrix} B_0^t & 0 \\ 0 & -B_2^t \end{pmatrix}, \quad JBJ = \begin{pmatrix} -B_2 & 0 \\ 0 & B_0 \end{pmatrix}.$$

If B and $B^{st} \text{ sign}$ are proportional, then one of the matrices B_0, B_2 is symmetric and the other one skew-symmetric. If B and JBJ are proportional, then the matrices B_0 and B_2 are proportional. This holds simultaneously only if $B_0 = 0$ and $B_2 = 0$.

If $p(B) = \bar{1}$, then

$$B = \begin{pmatrix} 0 & B_1 \\ B_3 & 0 \end{pmatrix}, \quad B^{st} \text{ sign} = \begin{pmatrix} 0 & B_3^t \\ B_1^t & 0 \end{pmatrix}, \quad JBJ = \begin{pmatrix} 0 & B_3 \\ -B_1 & 0 \end{pmatrix}$$

implying $B_1 = B_3 = 0$.

Therefore, if \mathfrak{g} is Q -irreducible, then $\mathcal{B} = 0$. □

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